UNIT 1

INTRODUCTION AND PRESTUDY FOR OPERATIONS RESEARCH

CONTAINING :

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WHAT IS OPERATIONS RESEARCH ?

1.1 INTRODUCTION : THE HISTORICAL DEVELOPMENT

In order to understand 'what Operations Research (OR)* is today,' we must know something of its history and evolution. The main origin of Operations Research was during the Second World–War. At that time, the military management in England called upon a team of scientists to study the strategic and tactical problems related to air and land defence of the country. Since they were having very limited military resources, it was necessary to decide upon the most effective utilization of them, *e.g.* the efficient ocean transport, effective bombing, etc.

During World-War II, the Military Commands of U.K. and U.S.A. engaged several inter-disciplinary teams of scientists to undertake scientific research into strategic and tactical military operations. Their mission was to formulate specific proposals and plans for aiding the Military Commands to arrive at the decisions on optimal utilization of scarce military resources and efforts, and also to implement the decisions effectively. The OR teams were not actually engaged in military operations and in fighting the war. But, they were only advisors and significantly instrumental in winning the war to the extent that the scientific and systematic approaches involved in OR provided a good intellectual support to the strategic initiatives of the military commands. Hence OR can be associated with ***"an art of winning the war without actually fighting it"***.

As the name implies, 'Operations Research' (sometimes abbreviated OR) was apparently invented because the team was dealing with *research* on (military) *operations*. The work of this team of scientists was named as *Operational Research* in England.

The encouraging results obtained by the British OR teams quickly motivated the United States military management to start with similar activities. Successful applications of the U.S. teams included the invention of new fight patterns, planning sea mining and effective utilization of electronic equipment. The work of OR team was given various names in the United States : ***Operational Analysis, Operations Evaluation, Operations Research, Systems Analysis, Systems Evaluation, Systems Research and Management Science.*** The name Operations Research was and is the most widely used so we shall also use it here.

Following the end of war, the success of military teams attracted the attention of *Industrial* managers who were seeking solutions to their complex executive-type problems. The most common problem was : what methods should be adopted so that the total cost is minimum or total profits maximum ? The first mathematical technique in this field (called the *Simplex Method* of linear programming) was developed in 1947 by American mathematician, **George B. Dantzig**. Since then, new techniques and applications have been developed through the efforts and cooperation of interested individuals in academic institutions and industry both.

Today, the impact of OR can be felt in many areas. A large number of management consultimg firms are currently engaged in OR activities. Apart from military and business applications, the OR activities include transportation system, libraries, hospitals, city planning, financial institutions, etc. Many of the Indian industries making use of OR activity are : ***Delhi Cloth Mills, Indian Railways, Indian Airlines, Defence Organizations, Hindustan Lever, Tata Iron & Steel Co., Fertilizer Corporation of India***, etc.

In business and other organizations, OR scientists and specialists always remain enagaged in the background. But, they help the top management officials and other line managers in doing their 'fighting' job better.

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* The short word **'OR'** for **'Operations Research'** should not be confused with the word **'or'** throughout the book.

While making use of the techniques of OR, a mathematical model of the problem is formulated. This model is actually a simplified representation of the problems in which only the most important features are considered for reasons of simplicity. Then, an *optimal* or *most favourable* solution is found. Since the model is an idealized form of exact representation of real problem, the optimal solution thus obtained may not prove to be the best solution to the actual problem. Although, extremely accurate but highly complex mathematical models can be developed, but they may not be easily solvable. So from both the cost-minimising and mathematical simplicity point of view, it seems beneficial to develop a less accurate but simpler model, and to find a sequence of solutions consisting of a series of increasingly better approximations to the actual course of action. Thus, the apparent weaknesses in the initial solution are used to suggest improvements in the model, its input-data, and the solution procedure. A new solution is thus obtained and the process is repeated until the further improvements in the succeeding solutions become so small that it does not seem economical to make further improvements.

If the model is carefully formulated and tested, the resulting solution should reach to be good approximation to the ideal course of action for the real problem. Although, we may not get the best answers, but definitely we are able to find *the bad answers where worse exist.* Thus operations research techniques are always able to save us from worse situations of practical life.

Q. 1. Comment the following statements : **[Rewa (Maths.) 93]**
- (i) O.R. is the art of winning war without actually fighting it.
- (ii) O.R. is the art of finding bad answers where worse exist.

2. What is O.R. ? **[Garhwal 97, 96; Meerut (IPM) 90]**

3. Enumerate six applications of Operations Research (O.R.) and describe one briefly. **[IGNOU 2001 (June)]**

1.2 THE NATURE AND MEANING OF 'OR'

[IPM (PGDBA)* 82, 81; Meerut (Math.) 82]

'OR' has been defined so far in various ways and it is perhaps still too young to be defined in some authoritative way. So it is important and interesting to give below a few opinions about the definition of OR which have been changed according to the development of the subject.

1. OR is a scientific method of providing executive departments with a *quantitative basis* for *decision* regarding the operations under their control. ***—Morse and Kimbal (1946)***
2. OR is a scientific method of providing executive with an *analytical* and *objective* basis for decisions. ***—P.M.S. Blackett (1948)***
3. The term 'OR' has hitherto-fore been used to connate various attempts to study operations of war by scientific methods. From a more general point of view, OR can be considered to be an attempt to study those *operations of modern society which involved organizations of men or of men and machines.* ***—P.M. Morse (1948)***
4. OR is the application of *scientific methods, techniques* and *tools* to problems involving the *operations of systems* so as to provide these in control of the operations with *optimum solutions* to the problem. ***—Churchman, Acoff, Arnoff (1957)***
5. OR is the art of giving *bad answers* to problems to which otherwise *worse answers* are given. ***—T. L. Saaty (1958)***
6. OR is a management *activity* pursued in two complementary ways—one half by the free and bold *exercise of commonsense* untrammelled by any routine, and other half by the application of a repertoire of *well established precreated methods* and techniques. ***—Jagjit Singh (1968)***
7. OR is the *attack* of modern methods on *complex problems* arising in the *direction and management* to large systems of men, machines, materials, and money in industry, business and defence. The distinctive approach is to developed a *scientific model* of the system, incorporating measurements of factors such as

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* Wherever the name of the examination is not mentioned in the University Examination references, it should be understood M.A./M.Sc. throughout the book.

* IPM = Institute of Productivity Management. *PGDBA* = Post-Graduate Diploma in Business Administration.

* The symbol Q. will stand for 'EXAMINATION QUESTIONS' throughout the book.

chance and risk with which to predict and *compare* the outcomes of alternative *decisions, strategies* or *controls*. The purpose is to help management to determine its policy and actions scientifically.

***—Operations Research Quarterly* (1971)**

8. Operations Research is the art of winning war without actually fighting it.
9. OR is an applied decision theory. It uses any *scientific mathematical or logical means* to attempt to cope with the problems that confront the executive when he tries to achieve a through going rationality in dealing with his decision problems. ***—Miller and Starr.***
10. OR is a scientific approach to problem solving for executive management. ***—H.M. Wagner***
11. OR is an aid for the executive in making his decisions by providing him with the needed quantitative information based on the scientific method of analysis. ***—C. Kittel***
12. OR is the systematic method oriented study of the basic structure, characteristics, functions and relationships of an organization to provide the executive with a sound, scientific and quantitative basis for decision making. ***—E.L. Arnoff & M.J. Netzorg***
13. OR is the application of *scientific methods* to problems arising from operations involving *integrated* systems of *men, machines and materials.* It normally utilizes the knowledge and skill of an inter-disciplinary research team to provide the managers of such systems with optimum operating solutions. ***—Fabrycky and Torgersen***
14. OR is an experimental and applied science devoted to observing, understanding and predicting the behaviour of purposeful man–machine systems and OR workers are actively engaged in applying this knowledge to practical problems in business, government, and society. ***—OR Society of America***
15. OR is the application of scientific method by inter-disciplinary teams to problems involving the controls of organized (man-machine) systems so as to provide solutions which *best serve the purpose of the organization as a whole.* ***—Ackoff & Sasieni, (1968)***
16. OR utilizes the planned approach (*updated scientific method)* and an *inter-disciplinary* team in order to represent complex functional relationships as mathematical models for purpose of providing a *quantitative basis* for decision making and *uncovering new problems* for quantitative analysis.

***—Thieanf and Klekamp* (1975)**

Comments on Definitions of OR :

From all above opinions, we arrive at the conclusion that whatever else 'OR' may be, it is certainly concerned with optimization problems. ***A decision, which taking into account all the present circumstances can be considered the best one, is called an optimal decision.*** **(Note)**

There are three main reasons for why most of the definitions of Operations Research are not satisfactory.

(i) First of all, Operations Research is not a science like any well-defined physical, biological, social phenomena. While chemists know about atoms and molecules and have theories about their interactions; and biologists know about living organisms and have theories about vital processes, *operations researchers* do not claim to know or have theories about operations. Operations Research is not a scientific research into the control of operations. It is essentially a collection of mathematical techniques and tools which in conjunction with a system approach are applied to solve practical decision problems of an *economic* or *engineering* nature. Thus it is very difficult to define Operations Research precisely.

(ii) Operations Research is inherently inter-disciplinary in nature with applications not only in military and business but also in medicine, engineering, physics and so on. Operations Research makes use of experience and expertise of people from different disciplines for developing new methods and procedures. Thus, inter-disciplinary approach is an important characteristic of Operations Research which is not included in most of its definitions. Hence most of the definitions are not satisfactory.

(iii) Most of the definitions of Operations Research have been offered at different times of development of 'OR' and hence are bound to emphasise its only one or the other aspect.

For example, 8th of the above definitions is only concerned with war alone. First definition confines 'OR' to be a scientific methodology applied for making operational decisions. It has no concern about the characteristics of different operational decisions and has not described how the scientific methods are applied in complicated situations. Many more definitions have been given by various authors but most of them fail to consider all basic characteristics of 'OR'. However, with further development of 'OR' perhaps more precise definitions should be forthcoming.

Q. 1. (a) Give any three definitions of Operations Research and explain them.
[J.N.T.U. (B. Tech.) 2003; Meerut (IPM) 91; Meerut (O.R.) 90]

(b) Give reasons for : why most of the definitions of Operations Research are not satisfactory.

2. Discuss the three approaches of MIS development. **[CA (May) 2000]**

3. What are the pre-requisites of a computer based MIS ? **[MCI 2000]**

1.3 MANAGEMENT APPLICATIONS OF OPERATIONS RESEARCH

Some of the areas of management decision making, where the 'tools' and 'techniques' of O R are applied, can be outlined as follows :

1. Finance–Budgeting and Investments
- (i) Cash-flow analysis, long range capital requirements, dividend policies, investment portfolios.
- (ii) Credit policies, credit risks and delinquent account procedures.
- (iii) Claim and complaint procedures.

2. Purchasing, Procurement and Exploration
- (i) Rules for buying, supplies and stable or varying prices.
- (ii) Determination of quantities and timing of purchases.
- (iii) Bidding policies.
- (iv) Strategies for exploration and exploitation of raw material sources.
- (v) Replacement policies.

3. Production Management
- (i) ***Physical Distribution***
  - (a) Location and size of warehouses, distribution centres and retail outlets.
  - (b) Distribution policy.
- (ii) ***Facilities Planning***
  - (a) Numbers and location of factories, warehouses, hospitals etc.
  - (b) Loading and unloading facilities for railroads and trucks determining the transport schedule.
- (iii) ***Manufacturing***
  - (a) Production scheduling and sequencing.
  - (b) Stabilization of production and employment training, layoffs and optimum product mix.
- (iv) ***Maintenance and Project Scheduling***
  - (a) Maintenance policies and preventive maintenance.
  - (b) Maintenance crew sizes.
  - (c) Project scheduling and allocation of resources.

4. Marketing
- (i) Product selection, timing, competitive actions.
- (ii) Number of salesman, frequency of calling on accounts per cent of time spent on prospects.
- (iii) Advertising media with respect to cost and time.

5. Personnel Management
- (i) Selection of suitable personnel on minimum salary.
- (ii) Mixes of age and skills.
- (iii) Recruitment policies and assignment of jobs.

6. Research and Development
- (i) Determination of the areas of concentration of research and development.
- (ii) Project selection.
- (iii) Determination of time cost trade-off and control of development projects.
- (iv) Reliability and alternative design.

From all above areas of applications, we may conclude that OR can be widely used in taking timely management decisions and also used as a corrective measure. The application of this tool involves certain data and not merely a personality of decision maker, and hence we can say : ***OR has replaced management by personality.***

Q. 1. "Operations Research replaces Management by personality." Discuss.
2. Explain applications of O.R. in Industry. **[Garhwal 97; Karnataka 95]**
3. Describe the various approaches used for development of MIS. **[MCI 2000]**

1.4 MODELLING IN OPERATIONS RESEARCH

Definition. *A model in the sense used in O R is defined as a representation of an actual object or situation. It shows the relationships (direct or indirect) and inter-relationships of action and reaction in terms of cause and effect.*

Since a model is an abstraction of reality, it thus appears to be less complete than reality itself. For a model to be complete, it must be a representative of those aspects of reality that are being investigated.

The main objective of a model is to provide means for analysing the behaviour of the system for the purpose of improving its performance. Or, if a system is not in existence, then a model defines the ideal structure of this future system indicating the functional relationships among its elements. The reliability of the solution obtained from a model depends on the validity of the model in representing the real systems. A model permits to 'examine the behaviour of a system without interfering with ongoing operations.

Models can be classified according to following characteristics :

1. Classification by Structure

(i) ***Iconic models.*** *Iconic models* represent the system as it is by scaling it up or down (*i.e.*, by enlarging or reducing the size). In other words, it is an image.

For example, a toy airplane is an iconic model of a real one. Other common examples of it are : photographs, drawings, maps etc. A model of an atom is scaled up so as to make it visible to the naked eye. In a globe, the diameter of the earth is scaled down, but the globe has approximately the same shape as the earth, and the relative sizes of continents, seas, etc., are approximately correct.

The iconic model is usually the simplest to conceive and the most specific and concrete. Its function is generally descriptive rather than explanatory. Accordingly, it cannot be easily used to determine or predict what effects many important changes on the actual system.

(ii) ***Analogue models.*** The models, in which one set of properties is used to represent another set of properties, are called *analogue models.* After the problem is solved, the solution is reinterpreted in terms of the original system.

For example, graphs are very simple analogues because distance is used to represent the properties such as : *time, number, per cent, age, weight;* and many other properties. Contour-lines on a map represent the rise and fall of the heights. In general, analogues are less specific, less concrete but easier to manipulate than are iconic models.

(iii) ***Symbolic (Mathematical) models.*** The *symbolic or mathematical model* is one which employs a set of mathematical symbols (*i.e., letters, numbers,* etc.) to represent the decision variables of the system. These variables are related together by means of a mathematical equation or a set of equations to describe the behaviour (or properties) of the system. The solution of the problem is then obtained by applying well-developed mathematical techniques to the model.

The symbolic model is usually the easiest to manipulate experimentally and it is most general and abstract. Its function is more often explanatory rather than descriptive.

2. Classification by Purpose

Models can also be classified by purpose of its utility. *The purpose of a model may be* descriptive, predictive or prescriptive.

(i) ***Descriptive models.*** A descriptive model simply describe some aspects of a situation based on observations, survey, questionnaire results or other available data. The result of an opinion poll represents a descriptive model.

(ii) ***Predictive models.*** Such models can answer 'what if' type of questions, *i.e.* they can make predictions regarding certain events. For example, based on the survey results, television networks such models attempt to explain and predict the election results before all the votes are actually counted.

(iii) ***Prescriptive models.*** Finally, when a predictive model has been repeatedly successful, it can be used to prescribe a source of action. For example, linear programming is a prescriptive (or normative) model because it prescribes what the managers ought to do.

3. Classification by Nature of Environment

These are mainly of two types :

(i) ***Deterministic models.*** Such models assume conditions of complete certainty and perfect knowledge. For example, linear programming, transportation and assignment models are deterministic type of models.

(ii) ***Probabilistic (or Stochastic) models.*** These types of models usually handle such situations in which the consequences or payoff of managerial actions cannot be predicted with certainty. However, it is possible to forecast a pattern of events, based on which managerial decisions can be made. For example, insurance companies are willing to insure against risk of fire, accidents, sickness and so on, because the pattern of events have been compiled in the form of probability distributions.

4. Classification by Behaviour

(i) ***Static models.*** These models do not consider the impact of changes that takes place during the planning horizon, *i.e.* they are independent of time. Also, in a static model only one decision is needed for the duration of a given time period.

(ii) ***Dynamic models.*** In these models, *time* is considered as one of the important variables and admit the impact of changes generated by time. Also, in dynamic models, not only one but a series of interdependent decisions is required during the planning horizon.

5. Classification by Method of Solution

(i) ***Analytical models.*** These models have a specific mathematical structure and thus can be solved by known analytical or mathematical techniques. For example, a *general linear programming* model as well as the specially structured *transportation* and *assignment models are analytical models.*

(ii) ***Simulation models.*** They also have a mathematical structure but *they cannot be solved* by purely using the 'tools' and 'techniques' of mathematics. A simulation model is essentially computer assisted experimentation on a mathematical structure of a real time structure in order to study the system under a variety of assumptions.

Simulation modelling has the advantage of being more flexible than mathematical modelling and hence can be used to represent complex systems which otherwise cannot be formulated mathematically. On the other hand, simulation has the disadvantage of not providing general solutions like those obtained from successful mathematical models.

6. Classification by Use of Digital Computers

The development of the digital computer has led to the introduction of the following types of modelling in OR.

(i) ***Analogue and Mathematical models combined.*** Sometimes analogue models are also expressed in terms of mathematical symbols. Such models may belong to both the types (ii) and (iii) in **classification 1** above.

For example, **simulation model** is of analogue type but mathematical formulae are also used in it. Managers very frequently use this model to '*simulate*' their decisions by summarizing the activities of industry in a scale-down period.

(ii) ***Function models.*** Such models are grouped on the basis of the function being performed.

For example, a function may serve to acquaint to scientist with such things as–tables, carrying data, a blue-print of layouts, a program representing a sequence of operations (like in computer programming).

(iii) ***Quantitative models.*** Such models are used to measure the observations.

For example, degree of temperature, yardstick, a unit of measurement of length value, etc.

Other examples of quantitative models are : (i) *transformation models* which are useful in converting a measurement of one scale to another (*e.g.*, Centigrade *vs* Fahrenheit conversion scale), and (ii) the *test models* that act as 'standards' against which measurements are compared (*e.g.*, business dealings, a specified standard production control, the quality of a medicine).

(iv) ***Heuristic models.*** These models are mainly used to explore alternative strategies (courses of action) that were overlooked previously, whereas mathematical models are used to represent systems possessing well-defined strategies. Heuristic models do not claim to find the best solution to the problem.

Q. 1. Model building is the essence of the 'O.R. approach'. Discuss.

2. Discuss in detail the three types of models with special emphasis on the important logical properties and the relationship the three types bear to each other and to modelled entities. **[Meerut (OR) 90]**

3. What is meant by a mathematical model of real situation ? Discuss the importance of models in the solution of Operational Research probems ? **[Bhuwneshwar 2004]**

4. What is a model ? Discuss various classification schemes of models. **[Agra 95, 94; C.A. (May) 92; Meerut (IPM) 90]**

1.5 PRINCIPLES OF MODELLING

Let us now outline general principles useful in guiding to formulate the models within the context of O R. The model building and their users both should be consiously aware of the following ***Ten*** principles :

1. ***Do not build up a complicated model when simple one will suffice.*** Building the strongest possible model is a common guiding principle for mathematicians who are attempting to extend the theory or to develop techniques that have wide applications. However, in the actual practice of building models for specific purposes, the best advice is to "keep it simple".
2. ***Beware of molding the problem to fit the technique.*** For example, an expert on linear programming techniques may tend to view every problem he encounters as required in a linear programming solutions. In fact, not all optimization problems involve only linear functions. Also, not all OR problems involve optimization. As a matter of fact, not all real-world problems call for *operations research* ! Of course, every one search reality in his own terms, so the field of OR is not unique in this regard. Being human, we rely on the methods we are most comfortable in using and have been most successful within the past. We are certainly not able to use techniques in which we have no competence, and we cannot hope to be competent in all techniques. We must divide OR experts into three main categories :
   (i) Technique developers, (ii) Teachers, and (iii) Problem solvers.
   In particular, one should be ready to tolerate the behaviour "I have found a cure but I am trying to search a disease to fit it" among *technique developers* and *teachers.*
3. ***The deduction phase of modelling must be conducted rigorously.*** The reason for requiring rigorous deduction is that one wants to be sure that if model conclusions are inconsistent with reality, then the defect lies in the assumptions. One application of this principle is that one must be extremely careful when programming computers. Hidden "bugs" are specially dangerous when they do not prevent the program from running but simply produce results which are not consistent with the intention of the model.
4. ***Models should be validated prior to implimentation.*** For example, if a model is constructed to forecast the monthly sales of a particular commodity, it could be tested using historical data to compare the forecasts it would have produced to the actual sales. In case, if the model cannot be validated prior to its implementation, then it can be implemented in phases for validation. For example, a new model for inventory control may be implemented for a certain selected group of items while the older system is retained for the majority of remaining items. If the model proves successful, more items can be placed within its range. It is also worthnoting that real things change in time. A highly satisfactory model may very well degrade with age. So periodic re-evaluation is necessary.
5. ***A model should never be taken too literally.*** For example, suppose that one has to construct an elaborate computer model of Indian economy with many competent researchers spending a great deal of time and money in getting all kinds of complicated interactions and relationships. Under such circumstances, it can be easily believed as if the model duplicates itself the real system. This danger continues to increase as the models become larger and more sophisticated, as they must deal with increasingly complicated problems.
6. ***A model should neither be pressed to do, nor criticized for failing to do that for which it was never intended.*** One example of this error would be the use of forecasting model to predict so far into the future that the data on which the forecasts are based have no relevance. Another example is the use of certain network methods to describe the activities involved in a complex project. A model should not be stretched beyond its capabilities.
7. ***Beware of over-selling a model.*** This principle is of particular importance for the OR professional because most non-technical benefactors of an operations researcher's work are not likely to understand his methods. The increased technicality of one's methods also increases the burden of responsibility on the OR. professional to distinguish clearly between his role as model manipulator and model interpreter. In those cases where the assumptions can be challenged, it would be dishonest to use the model.
8. ***Some of the primary benefits of modelling are associated with the process of developing the model.*** It is true in general that a model is never as useful to anyone else as it is to those who are involved in building it up. The model itself never contains the full knowledge and understanding of the real system that the builder must acquire in order to successfully model it, and there is no practical way to convey this knowledge and understanding properly. In some cases, the sole benefits may occur while the model is being developed. In such cases, the model may have no further value once it is completed. An example of

this case might occur when a small group of people attempts to develop a formal plan for some subject. The plan is the final model, but the real problem may be to agree on 'what the objectives ought to be'.

9. ***A model cannot be any better than the Information that goes into it.*** Like a computer program, a model can only manipulate the data provided to it; it cannot recognize and correct for deficiencies in input. Models may *condense* data or *convert* it to more useful forms, but they do not have the capacity to generate it. In some situations it is always better to gather more information about the system instead of exerting more efforts on modern constructions.
10. ***Models cannot replace decision makers.*** The purpose of OR models should not be supposed to provide "Optimal solutions" free from human subjectivity and error. OR models can aid decision makers and thereby permit better decisions to be made. However, they do not make the job of decision making easier. Definitely, the role of experience, intuition and judgement in decision making is undiminished.

1.6 APPROXIMATIONS (SIMPLIFICATIONS) OF 'OR' MODLES

While constructing a model, two conflicting objectives usually strike in our mind :

(i) The model should be as accurate as possible.

(ii) It should be as easy as possible in solving.

Besides, the management must be able to understand the solution of the model and must be capable of using it. So the reality of the problem under study should be simplified to the extent when there is no loss of accuracy.

The model can be simplified by :

(i) omitting certain variable *(ii)* changing the nature of variables *(iii)* aggregating the variables *(iv)* changing the relationship between variables, and *(v)* modifying the constraints, etc.

1.7 GENERAL METHODS FOR SOLVING 'OR' MODLES

Generally, three types of methods are used for solving OR models.

Analytic Method. If the OR model is solved by using all the tools of classical mathematics such as : differential calculus and finite differences available for this task, then such type of solutions are called *analytic solutions.* Solutions of various inventory models are obtained by adopting the so called analytic procedure.

Iterative Method. If classical methods fail because of complexity of the constraints or of the number of variables, then we are usually forced to adopt an iterative method. Such a procedure starts with a trial solution and a set of rules for improving it. The trial solution is then replaced by the improved solution, and the process is repeated until either no further improvement is possible or the cost of further calculation cannot be justified.

Iterative method can be divided into three groups :

(a) After a finite number of repetitions, no further improvement will be possible.

(b) Although successive iterations improve the solutions, we are only guaranteed the solution as a limit of an infinite process.

(c) Finally, we include the *trial* and *error* method which, however, is likely to be lengthy, tedious, and costly even if electronic computers are used.

The Monte-Carlo Method. The basis of so called Monte-Carlo technique is random sampling of variable's values from a distribution of that variable. Monte-Carlo refers to the use of sampling methods to estimate the value of non-stochastic variables. The following are the main steps of Monte-Carlo method :

Step 1. In order to have a general idea of the system, we first draw a *flow diagram* of the system.

Step 2. Then, we take correct sample observations to select some suitable model for the system. In this step, we compute the probability distributions for the variables of our interest.

Step 3. We, then, convert the probability distributions to a cumulative distribution function.

Step 4. A sequence of random numbers is now selected with the help of random number tables.

Step 5. Next, we determine the sequence of values of variables of interest with the sequence of random numbers obtained in **step 4**.

Step 6. Finally, we construct some standard mathematical function to the values obtained in ***step 5.***

Q. 1. State the different types of models used in OR. Explain briefly the general methods for solving these O.R. models. **[Agra 95]**

2. Write briefly about the following :
(i) Iconic models (ii) Analogue models (iii) Mathematical models **[Meerut (MCA III) May 2000]**

3. Explain (i) Mathematical Models (ii) Functional Models. **[JNTU (B. Tech.) 2003]**

The following interesting example will make the above procedure clear.

Illustration of Monte-Carlo Technique

Example. *A bombing mission is sent to bomb an important factory, which is rectangular in shape and has the dimensions 250 by 500 feet. The bombers will drop 10 bombs altogether, from high altitude, all aimed at the geometric centre of the plant. We assume that the bombing run is made parallel to the long dimension of the plant, that the deviation of the impact point from the aiming point is normal with mean zero and standard deviation 200 feet in each dimension, and that these two deviations are independent random variables. Use Monte-Carlo sampling to estimate the expected number of bomb-hits, and compare your result with the exact value.*

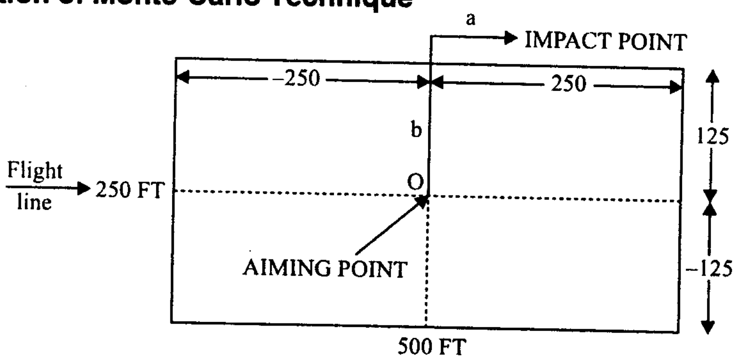


Fig. 1.1.

Solution. Let a be the horizontal deviation, and b be the vertical deviation, as shown in the Fig. 1.1.

The bomb will strike if the following conditions are satisfied :

$$-250 \le a \le 250, \quad -125 \le b \le 125, \quad \text{...(1·1)}$$

otherwise the bomb will miss the target.

If we put $x = a/200$, $y = b/200$, so that x and y will be the corresponding deviates read from a random number table, then the condition (1·1) for hitting the target becomes :

$$-250 \le 200x \le 250, \quad -125 \le 200y \le 125 \quad \text{...(1·2a)}$$

or

$$-1{\cdot}250 \le x \le 1{\cdot}250, \quad -0{\cdot}625 \le y \le 0{\cdot}625 \quad \text{...(1·2b)}$$

Results for the first three trials are given in Table 1·1 below.

Table 1·1

Bomb Trial 1, four hits	*x*	*y*	*Result*
1	– 0·291	1·221	Miss
2	– 2·828	– 0·439	Miss
3	0·247	1·291	Miss
4	– 0·584	0·541	**Hit***
5	– 0·446	– 2·661	Miss
6	– 2·127	0·665	Miss
7	0·656	0·340	**Hit***
8	1·041	0·008	**Hit***
9	0·899	0·110	**Hit***
10	– 1·114	1·297	Miss
Trial 2, two hits			
1	1·119	0·004	**Hit***
2	– 0·792	– 1·275	Miss
3	0·063	– 1·793	Miss
4	0·484	– 0·986	Miss
5	1·045	– 2·363	Miss
6	– 0·084	– 0·880	Miss
7	– 0·086	– 0·158	**Hit***
8	0·427	– 0·831	Miss
9	– 0·528	– 0·833	Miss
10	– 1·433	– 1·345	Miss

Trial 3, four hits	x	y	*Result*
1	− 2·015	− 0·594	Miss
2	− 0·623	− 1·047	Miss
3	− 0·699	− 1·347	Miss
4	0·481	0·996	Miss
5	0·586	− 1·023	Miss
6	0·579	0·551	**Hit***
7	0·120	0·418	**HIt***
8	0·191	0·074	**Hit***
9	0·071	0·524	**Hit***
10	− 3·001	0·479	Miss

These three trials give 3·33 as the average number of hits per mission. Many more trials should be conducted before we can have any real confidence in the result. One way of estimating : how many trials are necessary, is to list the cumulated mean at the end of each trial, and to stop the trials when the mean seems to have settled down to stable value. In this example, we have

after trial number	:	1	2	3
cummulated mean	:	4	3	3·33,

so that more trials are necessary.

The mean number of hits in a mission dropping 10 bombs is 3·69.

To compare the result with the exact value.

In this problem, unlike most Monte-Carlo problems, an exact calculation of the answer is much easier than the Monte Carlo calculation.

The probability of a hit with a single bomb is

$$\left[\int_{-1\cdot250}^{1\cdot250} f(x)\,dx\right] \times \left[\int_{-0\cdot625}^{0\cdot625} f(x)\,dx\right] = 2\cdot789 \times 0\cdot468 \text{ (from the table of the normal integers)}$$
$$= 0\cdot369.$$

Thus the previous value 3·69 is ten times of this value.

Advantages :

1. These methods avoid unnecessary expenses and difficulties that arise during the trial and error experimentation.
2. By this technique, we find the solution of much complicated mathematical expression which is not possible by any other method.

Disadvantages :

1. This technique does not give optimal answers to the problems. The good results are obtained only when the sample size is quite large.
2. The computations are much complicated even in simple cases.
3. It is a costly procedure for obtaining a solution of any related problem.

Q. 1. Write a short note on Monte-Carlo Technique and their usefulness in real life situations. **[Meerut (Stat.) 98]**

2. Describe the use of Monte-Carlo methods in sampling experiments. Illustrate with possible examples.

1.8 MAIN CHARACTERISTICS (FEATURES) OF OPERATIONS RESEARCH

The main characteristics of OR are as follows :

1. Inter-disciplinary team approach. In **OR**, the optimum solution is found by a team of scientists selected from various disciplines such as mathematics, statistics, economics, engineering, physics, etc.

For example, while investigating the inventory management in a factory, perhaps we may require an engineer who knows the functions of various items of stores. We also require a cost accountant and a mathematician-cum-statistician. Each member of such OR team is benefitted by the view points of others, so that the workable solution obtained through such collaborative study has a greater chance of acceptance by management.

Furthemore, an OR team required for a big organization may include a statistician, an economist, a mathematician, one or more engineers, a psychologist, and some supporting staff like computer programmers, etc. A mathematician or a probabilist can apply his tools in a plant problem only if he gets to understand some of the physical implications of the plant from an engineer. Otherwise, he may give such a solution which may not be possible to apply.

2. Wholistic approach to the system. The most of the problems tackled by OR have the characteristic that OR tries to find the *best* (*optimum*) decisions relative to largest possible portion of the total organization. The nature of organization is essentially immaterial.

For example, in attempting to solve a maintenance problem in a factory, OR tries to consider how this affects the production department as a whole. If possible, it also tries to consider how this effect on the production department in turn affects other department and the business as a whole. It may even try to go further and investigate how the effect on this particular business organization in turn affects the industry as a whole, etc. Thus OR attempts to consider inter-actions or chain of effects as far out as these effects are significant.

3. Imperfectness of solutions. By OR techniques, we cannot obtain perfect answers to our problems but, only the quality of the solution is improved from worse to bad answers.

4. Use of scientific research. OR uses techniques of scientific research to reach the optimum solution.

5. To optimize the total output. OR tries to optimize total return by maximizing the profit and minimizing the cost or loss.

Q. 1. Give the main characteristics of Operations Research. **[J.N.T.U. (B. Tech.) 2004, 03; C.A. (May) 92]**

2. Define OR and discuss its characteristics and limitations.

1.9 MAIN PHASES OF OPERATIONS RESEARCH STUDY

About fourty years ago, it would have been difficult to get a single operations-researcher to describe a procedure for conducting OR project. The procedure for an OR study generally involves the following major phases :

Phase I : Formulating the problem. Before proceeding to find the solution of a problem, first of all one must be able to formulate the problem in the form of an appropriate model. To do so, the following information will be required.

(i) Who has to take the *decision* ?

(ii) What are the *objectives* ?

(iii) What are the ranges of *controlled variables* ?

(iv) What are the uncontrolled variables that may affect the possible solutions ?

(v) What are the restrictions or constraints on the variables ?

Since wrong formulation cannot yield a right decision (solution), one must be considerably careful while execution this phase.

Phase II : Constructing a mathematical model. The second phase of the investigations is concerned with the reformulation of the problem in an appropriate form which is convenient for analysis. The most suitable form for this purpose is to construct a mathematical model representing the system under study. It requires the identification of both *static* and *dynamic* structural elements. A mathematical model should include the following three important basic factors :

(i) *Decision variables and parameters,* (ii) *Constraints or Restrictions,* (iii) *Objective function.*

Phase III : Deriving the solutions from the model. This phase is devoted to the computation of those values of decision variables that maximize (or minimize) the objective function. Such solution is called an *optimal solution* which is always in the best interest of the problem under consideration. The general techniques for deriving the solution of OR model are discussed in the following sections and further details are given in the text.

Phase IV : Testing the model and its solution (updating the model). After completing the model, it is once again tested as a whole for the errors if any. A model may be said to be valid if it can provide a reliable prediction of the system's performance. A good practitioner of Operations Research realises that his model be applicable for a longer time and thus he updates the model time to time by taking into account the past, present and future specifications of the problem.

Phase V : Controlling the solution. This phase establishes controls over the solution with any degree of satisfaction. The model requires immediate modification as soon as the controlled variables (one or more) change significantly, otherwise the model goes out of control. As the conditions are constantly changing in the world, the model and the solution may not remain valid for a long time.

Phase VI : Implementing the solution. Finally, the tested results of the model are implemented to work. This phase is primarily executed with the cooperation of Operations Research experts and those who are responsible for managing and operating the systems.

Q. 1. Discuss the various phases in solving an OR problem. **[IGNOU 2001; C.A. (Nov.) 92; Meerut (IPM) 90]**

2. What are various phases of O.R. problems ? Explain them briefly. **[VTU (BE Mech.) 2003]**

3. Give the different phases of Operations Research, and explain their significance in decision making. **[JNTU (B. Tech.) May 2004; Meerut (Stat.) 98, 90; Karnataka (B.E.) 95; C.A. (Nov.) 89]**

4. Explain the steps involved in the solution of an Operations Research problem. **[IGNOU 2001]**

5. What is an operations research? Discuss the various phases in solving an OR problem. **[AIMS (B.E.) Bangalore 2002]**

1.10 THE TERMS : 'TOOLS', 'TECHNIQUES' AND 'METHODS'

We now carefully differentiate the terms : 'tools', 'techniques' and 'methods' which are frequently used in science. It is evident that a table of random numbers is a *tool* of science. The way in which this tool is used is called a *technique*. The research plan which involves the use of ***Monte-Carlo*** procedure and the table of random numbers is called a *method* of science. Similarly, calculus is a scientific *tool*; employing calculus to find an optimum value of a variable in a mathematical model of a system is a scientific *technique*; and the plan of utilizing a mathematical model to optimize a system is a scientific *method*.

1·10-1 Scientific Method in Operations Research

The scientific method in OR study generally involves the three phases : (i) *the judgement phase*, (ii) *the research phase*, and (iii) *the action phase*.

Of these three, the *research phase* is the largest and longest, but the remaining two are just as important as they provide the basis for an implementation of the research.

The judgment phase includes :

(i) A determination of the operation.
(ii) The establishment of the objectives and values related to the operation.
(iii) The determination of the suitable measures of effectiveness.
(iv) Lastly, the formulation of the problems relative to the objectives.

The research phase utilizes :

(i) Observations and data collection for a better understanding of what the problem is.
(ii) Formulation of hypothesis and models.
(iii) Observation and experimentation to test the hypothesis on the basis of additional data.
(iv) Analysis of the available information and verification of the hypothesis using pre-established measures of effectiveness.
(v) Predictions of various results from the hypothesis, generalization of the result and consideration of alternative methods.

The action phase :

OR consists of making recommendations for decision process by those who first posed the problem for consideration, or by anyone in a position to make a decision influencing the operation in which the problem occured.

Q. 1. Discribe the phases of scientific method in Operations Research. **[JNTU (B. Tech.) 2003; Meerut (O.R.) 90]**

2. Enumerate the approach, technique and tools used in operations research. You may list as many as possible but focus on 4 tools and detail the appropriate computer hardware, software and application programs. **[IGNOU 2001, 99, 96]**

1.11 SCOPE OF OPERATIONS RESEARCH

In its recent years of organized development, OR has entered successfully many different areas of research for military, government and industry. The basic problem in most of the developing countries in Asia and Africa is to remove *poverty* and *hunger* as quickly as possible. So there is a great scope for economists, statisticians, administrators, politicians and the technicians working in a team to solve this problem by an OR approach. Besides this, OR is useful in the following various important fields.

1. In Agriculture. With the explosion of population and consequent shortage of food, every country is facing the problem of—

(i) optimum allocation of land to various crops in accordance with the climatic conditions; and
(ii) optimum distribution of water from various resources like canal for irrigation purposes.

Thus there is a need of determining best policies under the prescribed restrictions. Hence a good amount of work can be done in this direction.

2. In Finance. In these modern times of economic crisis, it has become very necessary for every government to have a careful planning for the economic development of her country. OR-techniques can be fruitfully applied :

(i) to maximize the per capita income with minimum resources;
(ii) to find out the profit plan for the company;
(iii) to determine the best replacement policies, etc.

3. In Industry. If the industry manager decides his policies (not necessarily optimum) only on the basis of his past experience (without using OR techniques) and a day comes when he gets retirement, then a heavy loss is encountered before the Industry. This heavy loss can immediately be compensated by newly appointing a young specialist of OR techniques in *business management*. Thus OR is useful to the *Industry Director* in deciding optimum allocation of various limited resources such as men, machines, material, money, time, etc., to arrive at the optimum decision.

4. In Marketing. With the help of OR techniques a *Marketing Administrator* (Manager) can decide :
(i) where to distribute the products for sale so that the total cost of transportation etc. is minimum,
(ii) the minimum per unit sale price,
(iii) the size of the stock to meet the future demand,
(iv) how to select the best advertizing media with respect to time, cost, etc.
(v) how, when, and what to purchase at the minimum possible cost ?

5. In Personnel Management. A personnel manager can use OR techniques :
(i) to appoint the most suitable persons on minimum salary,
(ii) to determine the best age of retirement for the employees,
(iii) to find out the number of persons to be appointed on full time basis when the workload is seasonal (not continuous).

6. In Production Management. A production manager can use OR techniques :
(i) to find out the number and size of the items to be produced;
(ii) in scheduling and sequencing the production run by proper allocation of machines;
(iii) in calculating the optimum product mix; and
(iv) to select, locate, and design the sites for the production plants.

7. In L.I.C. OR approach is also applicable to enable the L.I.C. offices to decide :
(i) what should be the premium rates for various modes of policies,
(ii) how best the profits could be distributed in the cases of with profit policies ? etc.

Finally, we can say : wherever there is a problem, there is OR. The applications of OR cover the whole extent of any thing. A recent publication of the OR society contains a summary of the applications of OR. The reader wishing more details on applications may consult the publication : ***'Progress in OR'*** **Vol. 2 by** ***Hertz., D.B.*** **and** ***R.T. Eddison.***

Q. 1. Define O.R. and discuss its scope. **[Meerut (Stat.) 98; Garhwal 96; Kanpur 96; Rewa (Maths.) 93; Rohil. 93, 92]**
2. What are the areas of applications of O.R., **[Meerut (Maths) 91]**
3. (a) Explain the meaning, scope and methodology of O.R. **[VTU (BE Mech.) 2002]**
(b) Discuss the significance and scope of Operations Research in modern management. **[Delhi Univ. (MBA) HCA 2001]**
4. Write a critical essay on the definition and scope of Operations Research. **[JNTU (B. Tech) 2003, 02; Virbhadrah 2000]**

1.12 ROLE OF OPERATIONS RESEARCH IN DECISION-MAKING

The Operations Research may be regardeed as a tool which is utilized to increase the effectiveness of management decisions. In fact, OR is the objective suppliment to the subjective feeling of the administrator (decision-maker). Scientific method of OR is used to understand and describe the phenomena of operating system. OR models explain these phenomena as to what changes take place under altered conditions, and control these predictions against new observations. For example, OR may suggest the best locations for factories, warehouses as well as the most economical means of transportation. In marketing, OR may help in indicating the most profitable type, use and size of advertising compaigns subject to the financial limitations.

The advantages of OR study approach in business and management decision making may be classified as follows:

1. **Better Control.** The management of big concerns finds it much costly to provide continuous executive supervisions over routine decisions. An OR approach directs the executives to devote their attention to more pressing matters. For example, OR approach deals with production scheduling and inventory control.

2. **Better Co-ordination.** Sometimes OR has been very useful in maintaining the law and order situation out of chaos. For example, an OR based planning model becomes a vehicle for coordinating marketing decisions with the limitations imposed on manufacturing capabilities.

3. **Better System.** OR study is also initiated to analyse a particular problem of decision making such as establishing a new warehouse. Later, OR approach can be further developed into a system to be employed repeatedly. Consequently, the cost of undertaking the first application may improve the profits.

4. **Better Decisions.** OR models frequently yield actions that do improve an intuitive decision making. Sometimes, a situation may be so complicated that the human mind can never hope to assimilate all the important factors without the help of OR and computer analysis.

In the present text, we restrict ourselves to discuss the problems on : ***Inventory control, Replacement, Queues, Linear programming, Goal Programming, Transportation, Assignment, Games theory, Sequencing, Dynamic programming, Information theory, PERT/CPM, Simulation, and Decision theory*** etc.

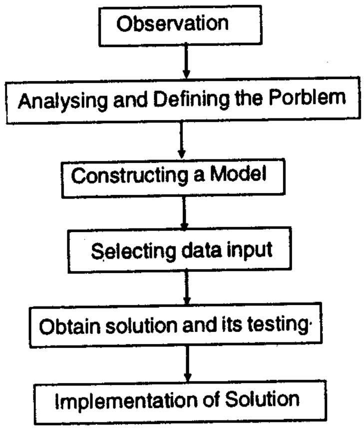
O.R. provides a logical and systematic approach for decision-making. The phases and process study must also be quite logical and systematic. There are six important steps in O.R. study, but in all each and every step docs not necessarily follow logical order as below :

Step I **: Observing the Problem Environment**

The activities in this step are visits, conferences, observations, research etc. With such activities analyst gets sufficient information and support to define the problem.

Step II **: Analysing and Defining the Problem**

In this step the problem is defined, and objectives and limitations of the study are stated in its context. One thus gets clear grasp of need for a solution and indication of its nature.



Step III **: Developing a Model**

Step III is to contruct a model. A model is representation of some real or abstract situations. O.R. models are basically mathematical models representing systems, processes or environment in the form of equations, relationships or formulae. The activities in this step are defining interrelationships among variables, formulating equations, using known O.R. models or searching suitable alternate models. The proposed model may be practically tested and modified in order to work under given environmental constraints. A model may also be managerial if not satisfied with the solution it offers.

***Step IV* : Selecting Appropriate Data Input**

No model will work appropriately if data input is not appropriate. Hence using right kind of data is vital in O.R. process, Important activities in this step are analysing internal-external data and facts, collecting openions and using computer data banks. The purpose of the step is to have sufficient input to operate and test the models.

***Step V* : Providing a Solution and Testing its Reasonableness**

Step V is performed to obtain a solution with the help of model and data input. Such a solution is not implemented immediately. First it is tested not behaving properly, updating and modification of the model is considered at this stage. The end result is solution that supports current organization objective.

***Step VI* : Implementing the Solution**

Implementation of the solution is the last step of O.R. process. In O.R., the decision-making is scientific, but implementation of decision involves many behavioural issues. Therefore, the implementing authority has to resolve the behavioural issues. He has to convince not only the workers but also the superiors. The gap between one who provides a solution and the other who wishes to use it has to be eliminated. To achieve this, O.R. analyst as well as management should play a positive role. Needless to say a properly implemented solution obtained through O.R. techniques results an improved working and gets active management support.

Q. 1. What is the importance (role) of Operations Research in decision making. **[Kanpur 96]**

2. Describe in brief the role of quantitative techniques in bussiness management. **[JNTU (B. Tech.) 2003]**

3. What are the various phases through which an O.R. team normally has to proceed ?

1.13 BRIEF OUTLINES OF OR-MODELS : QUANTITATIVE TECHNIQUES OF OR

A brief account of some of the important OR models is given below :

1. Distribution (Allocation) Models. Distribution models are concerned with the allotment of available resources so as to minimise cost or maximise profit subject to prescribed restrictions. Methods for solving such type of problems are known as *mathematical programming techniques.* We distinguish between linear and non-linear programming problems on the basis of linearity and non-linearity of the objective function and/or constraints respectively. In linear programming problems, the objective function is linear and constraints are also linear inequalities/equations . Transportation and Assignment models can be viewed as special cases of linear programming. These can be solved by specially devised procedures called *Transportation* and *Assignment Techniques.*

In case the decision variables in a linear programming problem are restricted to either integer or zero-one value, it is known as *Integer* and *Zero-One programming problems*, respectively. The problems having multiple, conflicting and incommensurable objective functions (goals) subject to linear constraints are called *linear goal programming problems.* If the decision variables in a linear programming problem depend on chance, then such problems are called *stochastic linear programming problems.*

2. Production/Inventory Models. Inventory/Production models are concerned with the determination of the optimal (economic) order quantity and ordering (production) intervals considering the factors such as–demand per unit time, cost of placing orders, costs associated with goods held up in the inventory and the cost due to shortage of goods, etc. Such models are also useful in dealing with quantity discounts and multiple products.

3. Waiting Line (or Queueing) Models. In queueing models an attempt is made to predict :

(i) how much average time will be spent by the customer in a queue ?

(ii) what will be an average length of waiting line or queue ?

(iii) what will be the traffic intensity of a queueing system ? etc.

The study of waiting line problems provides us methods to minimize the sum of costs of providing service and cost of obtaining service which are primarily associated with the value of time spent by the customer in a queue.

4. Markovian Models. These models are applicable in such situations where the state of the system can be defined by some descriptive measure of numerical value and where the system moves from one state to another on a probability basis. Brand-swiching problems considered in marketing studies is an example of such models.

5. Competetive Strategy Models (Games Theory). These models are used to determine the behaviour of decision-making under competition or conflict. Methods for solving such models have not been found suitable for industrial applications, mainly because they are referred to an idealistic world neglecting many essential features of reality.

6. Network Models. These models are applicable in large projects involving complexities and inter-dependencies of activities. *Project Evaluation and Review Techniques* (PERT) and *Critical Path Method* (CPM) are used for planning, scheduling and controlling complex project which can be characterised as net-works.

7. Job Sequencing Models. These models involve the selection of such a sequence of performing a series of jobs to be done on service facilities (machines) that optimize the efficiency measure of performance of the system. In other words, sequencing is cencerned with such a problem in which efficiency measure depends upon the order or sequence of performing a series of jobs.

8. Replacement Models. These models deal with the determination of optimum replacement policy in situations that arise when some items or machinery need replacement by a new one. Individual and group replacement policies can be used in the case of such equipments that fail completely and instantaneously.

9. Simulation Models. Simulation is a very powerful technique for solving much complex models which cannot be solved otherwise and thus it is being extensively applied to solve a variety of problems. This technique is more useful when following two types of difficulties may arise :

(i) The number of variables and constraint relationships may be so large that it is not computationally feasible to pursue such analysis.

(ii) 'Secondly, the model may be much away from the reality that no confidence can be placed on the computational results.

In fact, such models are solved by simulation techniques where no other method is available for its solution.

Operations Research, as its name suggests, gives stress on analysis of operations as a whole. For this purpose it uses any suitable techniques or tools available from the fields of mathematics, statistics, cost analysis or numerical calculations. Some such techniques are listed below :

(1) Linear Programming	(2) Non-linear Programming	(3) Integer Programming
(4) Dynamic Programming	(5) Goal Programming	(6) Games Theory
(7) Inventory Control	(8) PERT-CPM	(9) Simulation

(10) Queueing Theory etc. Here, for example, we describe in brief the *queueing* or *waiting line theory*.

Queues have become an integral part of our daily life. Queues are formed everywhere where a service is fered and the service rate is slower than the arrival rate of customers. People waiting for railway reservations, machines waiting for repairs at a workshop and aeroplanes waiting in the sky to find a place to land at the airport are all examples of queueing.

Costs are associated with the waiting in a line and costs are also associated with adding more service facilities or counters. The purpose of OR study is to decide the optimum number of service facilities so as to minimise the sum of waiting period cost and cost of providing facilities.

Queueing theory works out the expected number of people in the queue, expected waiting time in the queue, expected idle time for the server etc. These calculations then help in deciding the optimum number of service facilities under given constraints.

Q. 1. Write a note on application of various quantitative techniques in different fields of business decision making.

2. Explain various types of O.R. models and indicate their application to production, inventory, and distribution systems.

3. Enumerate six techniques of operations Research and describe one briefly. **[IGNOU 2001 (Jan)]**

1.14 DEVELOPMENT OF OPERATIONS RESEARCH IN INDIA

In 1949, Operations Research came into picture when an OR unit was established at the Regional Research Laboratory, Hyderabad. At the same time, Prof. R.S. Verma (Delhi University) setup an OR team in the Defence Science Laboratory to solve the problems of store, purchase and planning. In 1953, Prof. P.C. Mahalanobis established an OR team in the Indian Statistical Institute, Calcutta, for solving the problem of national planning and survey. In 1957, Operations Research Society of India was formed and this society became a member of the International Federation of Operations Research Societies in 1960. Presently India is publishing a number of research journals, namely, ***'OPSEARCH'***, ***'Industrical Engineering and***

Management', 'Materials Management Journal of India', 'Defence Science Journal', 'SCIMA', 'Journal of Engineering Production', etc.

As far as the OR education in India is concerned University of Delhi was the first to introduce a complete M.Sc. course in OR in 1963. Simultaneously, Institute of Management at Calcutta and Ahemdabad started teaching OR in their MBA courses. Now-a-days, OR has become so popular subject that it has been introduced in almost all Institutes and Universities in various disciplines like, Mathematics, Statistics, Commerce, Economics, Management Science, Medical science, Engineering, etc. Also, realizing the importance of OR in Accounts and Administration, government has introduced this subject for the IAS, CA, ICWA examinations, etc.

Prof. Mahalanobis first applied OR in India by formulating second five-year plan with the help of OR techniques. Planning Commission made the use of OR techniques for planning the optimum size of the Caravelle fleet of Indian air lines. Some of the industries, namely, ***Hindustan Lever Ltd.; Union Carbide, TELCO, Hindustan Steel, Imperial Chemical Industries, Tata Iron & Steel Company, Sarabhai Group, FCI***, etc. have engaged OR teams. ***Kirlosker Company*** is using the assignment technique of OR to maximize profit.

Textile firms like, DCM., Binni's and Calico, etc., are using linear programming techniques. Among other Indian organizations using OR are the **Indian Railways, CSIR, Tata Institute of Fundamental Research, Indian Institute of Science, State Trading Corporation**, etc.

****It is also worthnoting that the present text on* 'OPERATIONS RESEARCH' *is the first book published in India to meet the requirements of various courses on this subject.***

1.15 ROLE OF COMPUTERS IN OPERATIONS RESEARCH

In fact, computers have played a vital role in the development of OR. But OR would not have achieved its present position for the use of computers. The reason is that—in most of the OR techniques computations are so complex and involved that these techniques would be of no practical use without computers. Many large scale applications of OR techniques which require only few minutes on the computer may take weeks, months and sometimes years even to yield the same results manually. So the computer has become as essential and integral part of OR. Now-a-days, OR methodology and computer methodology are growing up simultaneously. It seems that in the near future the line dividing the two methodologies will disappear and the two sciences will combine to form a more general and comprehensive science. It should also be noted that FORTRAN and C-programs are functionally equivalent.

The computor software packages are useful for rapid and effective calculations which is a necessary part of O.R. approach to solve the problems. These are :

(i) QSB+ (Quantitative System for Business Plus), Version 3.0, by Yih-long Chang and Robert S. Sullivan, is a software package that contains problem solving algorithms for OR/MS, as well as modules on basic statistics, non-linear programming and financial analysis.

(ii) QSOM (Quantitative Systems for Operations Management), by Yih-long, is an interactive user-friendly system. It contains problem-solving algorithms for operations management problems and associated information system.

(iii) Value STORM : MS quantitative Modelling for Decision Support, by Hamilton Emmons, A.D. Flowers, Chander Shekhar, M.Khot and Kamlesh Mathur, is a special version of Personal STORM version 3.0 developed for use in OR/MS.

(iv) Excel 97 by Gene Weiss Kopf and distributed by BPB publications, New Delhi, is an easy-to-use task-oriented guide to Excel Spread sheet applications.

(v) LINDO (Linear Interactive Discrete Optimization), developed by Linus Schrage Lindo in his book "An Optimization Modeling System, 4th ed. (Palo Alto, CA : Scientific Press 1991)

SELF-EXAMINATION QUESTIONS

1. (a) What is Operations Research ? A certain wine importer noticed that his sales of wine were not what they should be in comparison to other types of liquor. He hired you as a consultant to look into this problem, with the intention of improving the wine business. What would you do ?

   (b) How does one go about organising for effective Operations Research ? Explain.
2. Give a brief account of the methods used in model formulation.
3. Explain, how and why OR methods have been valuable in aiding executive decision. **[Meerut (Stat.) 90]**
4. Explain the concept, scope and tools of OR as applicable to business and industry.
5. Discuss the advantages and limitations of using results from a mathematical model to make decisions about operations.

6. "Mathematics of OR is mathematics of optimization". Discuss.
7. "OR is the application of scientific methods, techniques and tools to problem involving the operations of a system so as to provide those in control of the system with optimum solutions to the problem." Discuss.
8. (a) Define Operations Research. Give the main characteristics of Operations Research.
   (b) Discuss the importance of Operations Research in decision-making process.
9. (a) Dicuss the significance and scope of Operations Research in modern management.
   (b) Describe in brief the uses of statistical techniques in Operations Research.
10. Write a detailed note on the use of models for decision-making. Your answers should specifically cover the following :
    (i) Need for model building. (ii) Type of model appropriate to the situation.
    (iii) Steps involved in the construction of the model. (iv) Setting up criteria for evaluating different alternatives.
    (v) Role of random numbers.
11. "Operations Research is an aid for the executive in making his decisions by providing him with the needed quantitative information, based on the scientific method analysis". Discuss this statement in detail, illustrating it with O.R methods that you know. **[Meerut (Stat.) 95]**
12. Give the essential characteristics of the following types of process :
    (a) Allocation (b) Competitive Games (c) Inventory (d) Waiting line.
13. What is the role of Operations Research in decision making ? Explain the scope and methodology of Operations Research, the main phases of Operations Research and techniques in solving an Operations Research problem.
14. Write short notes on the following :
    (i) Area of applications of Operations Research.
    (ii) Role of constraints and objectives in the construction of mathematical models.
    (iii) Statistician's role as member of O.R team.
15. Is Operations Research a discipline, or a profession, or set of techniques, or a philosophy, or a new name for an old thing ?
16. Outline broad features of the judgment phase and the research phase of scientific method in Operations Research. Discuss fully any one of these phases.
17. How can Operations Research models be classified ? Which is the best classification in terms of learning and understanding the fundamentals of Operations Research ?
18. What are the advantages and disadvantages of Operational Research models ? Why is it necessary to test models and how would you go about testing a model ?
19. What is Operations Research ? Describe four models used in Operations Research.
20. List any three Operations Research techniques and state in what conditions they can be used ?
21. Explain the role of quantitative techniques in the field of business and industry in modern times. Give a few examples in support of your answer.
22. What are the essential characteristics of Operations Research ? Mention different phases in an Operations Research study. Point out some limitations of Operations Research.
23. (a) Define OR as a decision making science.
    (b) Briefly explain the uses of OR-techniques in India. How are they found useful by the business executives ? Which of the three techniques are most commonly used in India ? Why
24. Explain the meaning and nature of OR.
25. State any four areas for the application of OR techniques in Financial Management, and how it improves the performance of the organisation.
26. (a) Comment on "Operations Research is a scientific and for enhancing creative and judicious capabilities of a decision maker".
    (b) Give any four processes of Operations Research and discuss their essential features.
27. Write a critical essay on the definition and scope of Operations Research.
28. Comment on the following statements :
    (a) OR is a bunch of mathematical techniques.
    (b) OR is no more than a quantitative analysis of the problem.
    (c) OR advocates a system's approach and is concerned with optimization. It provides a quantitative analysis for decision making.
    (d) OR has been defined semi-facetiously as the application of big minds to small problems.
29. What is Operations Research ? What areas of Operations Research have made a significant impact on decision making process ? Why is it important to keep an open mind in utilizing Operations Research techniques ?
30. Give a definition of Operations Research indicating the different types of models of the problem and the general methods of their solution.
31. **(a) Write briefly about the following :**
    (i) Iconic models, (ii) Analogue models, (iii) Mathematical models (or Symbolic models).

(b) Explain three types of models used in Operations Research, giving suitable example.

(c) What is the function of a model in decision making ? Name the types of models. What are the advantages of models? What are the pitfalls of models.

32. Distinguish the following models with suitable examples :

(i) Stochastic and deterministic models; (ii) Static and dynamic models.

33. Quantitative techniques complement the experience and judgement of an executive in decision making. They do not and cannot replace it. Discuss. **[Delhi Univ. (MBA) 1988]**

34. In construction any OR model, it is essential to realize that a most important purpose of the modelling process is "to help any manager better." Keeping this purpose in mind, state any four OR models that can be of help to Chartered Accountants in advising their clients. **[C.A. (May) 91]**

35. State three properties and three advantages of an OR model. **[C.A. (May) 92]**

36. Describe briefly the components of a proplem and mention the three major types of problems in decision making under different environment. **[C.A. (Nov.) 92]**

37. "Much of the success of OR applications in the last three decades is due to the computers." Discuss. **[C.A. (May) 93]**

38. Discuss the role and scope of quantitative methods for scientific decision making in a business environment. **[IPM (MBA) 2000]**

39. Explain briefly the various applications of O.R. **[VTU (BE Compu.) Aug. 2001]**

40. What are the advantages and limitations of OR studies ? **[VTU (BE VIth Sem.) Feb. 2002]**

41. What are the essential characteristics of operations research ? Mension different phases in the operations research study. Point out its limitations, if any. **[C.A. Nov 1992]**

42. (a) Why is the study of Operations Research important to the decision maker.

(b) Operation Research increases creative and judicious capabilities of a decision maker. Comment. **[Delhi Univ. (MBA) 1998]**

OBJECTIVE QUESTIONS

1. Operations research approach is
(a) multi-disciplinary. (b) scientific. (c) intuitive. (d) all of the above.

2. Operations research analysts do not
(a) predict future operations. (b) build more than one model.
(c) collect relevant data. (d) recommend decision and accept.

3. For analysing a problem, decision-makers should normally study
(a) its qualitative aspects. (b) its quantitative aspects. (c) both (a) and (b). (d) neither (a) nor (b).

4. Decision variables are
(a) controllable. (b) uncontrollable. (c) parameters. (d) none of the above.

5. A model is
(a) an essence of reality. (b) an approximation. (c) an idealization. (d) all of the above.

6. Managerial decisions are based on
(a) an evaluation of quantitative data. (b) the use of qualitative factors.
(c) numbers produced by formal models. (d) all of the above.

7. The use of decision models
(a) is possible when the variable's value is known.
(b) reduces the scope of judegement and intuition known with certainty in decision-making.
(c) requires the knowledge of computer software use.
(d) none of the above.

8. Every mathematical model
(a) must be deterministic. (b) requires computer aid for its solution.
(c) represents data in numerical form. (d) all of the above.

9. A physical model is example of
(a) an iconic model. (b) an analogue model. (c) a verbal model. (d) a mathematical model.

10. An optimization model
(a) mathematically provides the best decision. (b) provides decision within its limited context.
(c) helps in evaluating various alternatives constantly. (d) all of the above.

Answers

1. (a) **2.** (a) **3.** (c) **4.** (a) **5.** (d) **6.** (d) **7.** (d) **8.** (c) **9.** (a) **10.** (d).

***

2

PRE-STUDY FOR OPERATIONS RESEARCH

2.1 INTRODUCTION

The main purpose of this chapter is to provide with basic concepts of preliminary mathematics required for understanding this new subject of Operations Research. Due to wide scope of O R in various disciplines, some useful topics of elementary mathematics are discussed in this chapter. It covers some important concepts on : ***Vectors and Linear algebra, Matrices and determinants, Simultaneous linear equations, Differentiation and integration, Generating functions, Finite differences,*** etc.

The mathematical methods are presented in this chapter with a minimum mathematical complexity. However, this approach does not reduce the potential value of these methods.

I–Vectors and Linear Algebra

2.2 CONCEPT OF VECTORS

It is well-known that a point can be expressed in terms of an origin and coordinate axes. Each point in 2-dimension (or 2-space) is uniquely represented by an ordered pair of real numbers, known as the coordinates of the point. The pair is said to be ordered because, *e.g.* the point represented by (0, 1) is not the same as the point (1, 0). Also, each point uniquely represents a line from the origin to the point. This line will be called a ***vector.***

As explained in Fig. 2·1, the vector has ***direction*** (from origin to the point) and ***magnitude*** (the length of the line). Thus, there exists a unique correspondence between vectors directed from the origin and ordered pairs of real numbers in 2–dimensional space.

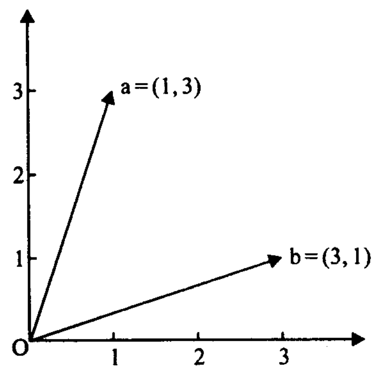


Fig. 2.1.
Vectors in 2–space.

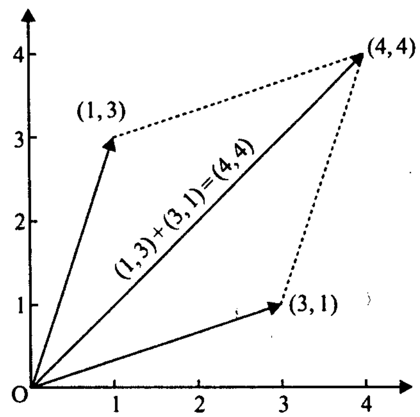


Fig. 2.2
Sum of vectors

In 2–space, the sum of two vectors —*e.g.*, (1, 3) and (3, 1)—is given by the sum of corresponding elements (1 + 3, 3 + 1) = (4, 4) which is represented by a parallelogram in Fig. 2·2.

These concepts of vectors can be extended to *n*–dimensional spaces as follows.

2.3 DEFINITIONS

Vector. A *vector* in n-space is an ordered set of n-real numbers.

For example, $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is a vector of elements or components. The real numbers $a_1, a_2, \ldots, a_n$ are called the components of **a**.

Equality of two n-vectors. Two n-vectors $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ are said to be ***equal*** if respective components of **a** and **b** are equal, *i.e.*, $a_1 = b_1$, $a_2 = b_2, \ldots, a_n = b_n$.

Also, note that $\mathbf{a} = \mathbf{b} \Leftrightarrow \mathbf{b} = \mathbf{a}$.

Sum and difference of two vectors. The *sum* of two n-vectors $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ is defined to be the n-vector **c** given by

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) = \mathbf{c}.$$

The subtraction of **b** from **a**, is defined to be the n-vector **d** given by

$$\mathbf{a} - \mathbf{b} = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n) = \mathbf{d}.$$

For given n-vectors **a**, **b** and **c**, we have

(i) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$, *(ii)* $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.

Multiplication of a vector by scalar. The product of an n-vector $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ by a scalar λ, positive or negative is defined to be the n-vector as,

$$\lambda\mathbf{a} = \lambda (a_1, a_2, \ldots, a_n) = (\lambda a_1, \lambda a_2, \ldots, \lambda a_n)$$

In general, for the given n-vectors **a** and **b** and the scalars λ and μ, we have

(i) $\lambda (\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$, *(ii)* $\lambda (\mu\mathbf{a}) = (\lambda\mu)\, \mathbf{a}$.

Because we discuss multiplication of vectors, it will be necessary to distinguish between a ***row vector*** and a ***column vector.***

Row-vector and column-vector. An n-vector $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is called a ***row-vector.*** Corresponding to each row-vector $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, there corresponds another n-vector written as

$$\mathbf{a}^T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ or simply by } [a_1 \;\; a_2 \;\; \ldots \;\; a_n]^T, \text{ for convenience,}$$

which is called a ***column vector.*** There is no conceptual difference between them ; it is purely a matter of usage.

Note. Since row-vector and column-vector both are n-vectors, these two types are different. So a row-vector can be added to a row-vector only. Similar will be the case for column vectors. We will assume that *all vectors are column vectors* unless otherwise stated (but for convenience they will be written as row vectors.)

If **a** is a column-vector, we can denote its row-vector equivalent as $\mathbf{a}^T$.

Inner-product or scalar-product. The ***inner product*** of two n-vectors **a** and **b**, written **a•b** or simply **ab** is a number given by

$$\mathbf{ab} = a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n.$$

Scalar-product is the useful multiplication of vectors. It is called the scalar-product because the result of the multiplication is *scalar* (a real number—not a vector).

For example, if $\mathbf{a} = (1\ 2\ 3)$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, then

$$\mathbf{ab} = (1\ 2\ 3) \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, = 4 + 10 + 18 = 32.$$

Note that in the inner-product **ab**, vector **a** is treated as row-vector and vector **b** is treated as column-vector. If there are no chances of any confusion, for economy of space, we may write both **a** and **b** in the form of rows only. In above example, we may write

$$\mathbf{a} \bullet \mathbf{b} = (1\ 2\ 3) \bullet [4\ 5\ 6] = 4 + 10 + 18 = 32\,.$$

In general, for the given vectors **a**, **b**, **c** and scalar λ, we have (*i*) **ab** = **ba**, (*ii*)**a**(**b** + **c**) = **ab** + **ac**, (*iii*) (**a** + **b**)**c** = **ac** + **bc**, (*iv*) **a**(λ**b**) = λ (**ab**), (*v*) (λ**a**)**b** = λ (**ab**).

Euclidean space (E^n). An n-dimensional space is called ***Euclidean space*** (denoted by E^n); 2-space (E^2) and 3-space (E^3) are the special cases of it.

Distance between two points. In n-space E^n, the distance between $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ is

$$|\mathbf{a} - \mathbf{b}| = \sqrt{[(a_1 - b_1)^2 + (a_2 - b_2)^2 + \ldots + (a_n - b_n)^2]}.$$

Length of a vector. The distance between two points gives the ***length of a vector.*** The length of a vector is the distance between origin and the point identifying the vector.

The length of a vector $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ in E^n is given by $|\mathbf{a}| = \sqrt{[a_1^2 + a_2^2 + \ldots + a_n^2]}$.

2.4 SPECIAL VECTORS

Special names are given to the commonly used vectors. There are—

Null vector. The null vector is a vector whose elements are all zero : $\mathbf{0} = (0, 0, \ldots, 0)$.

The null vector corresponds to the origin.

Sum vector. The sum vector is the vector whose elements are all 1 : $\mathbf{1} = (1, 1, \ldots, 1)$.

Unit vector (e_i). The unit vector e_i is the vector whose i th element is 1 and all other elements are 0.

Examples of unit vectors are $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0)$, etc. In E^2, there are two unit vectors while in E^n, there are n unit vectors. Geometrically these vectors lie along the axes of the space and have unit length.

Orthogonal vectors. Two vectors **a** and **b** are said to be orthogonal, if **a.b** = 0.

2.5 VECTOR INEQUALITIES

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$. Then **a** is said to be ***greater than or equal to*** **b,** if and only if, $a_1 \geq b_1, a_2 \geq b_2, \ldots, a_n \geq b_n$. This is denoted by $\mathbf{a} \geq \mathbf{b}$.

Similarly, if $a_1 \leq b_1, a_2 \leq b_2, \ldots, a_n \leq b_n$, then **a** is said to be ***less than or equal to*** **b,** denoted by $\mathbf{a} \leq \mathbf{b}$.

Two vectors **a** and **b** are said to be ***order comparable***, if and only if, they have equal number of elements.

Example. Let

$\mathbf{a} = (1, 2, 3), \mathbf{b} = (0, 1, 2)$. Clearly, $\mathbf{a} \geq \mathbf{b}$.

2.6 LINEAR COMBINATION OF VECTORS

Suppose we are given two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. The operations of addition and scalar multiplication can be applied to obtain the expressions like $\mathbf{a}_1 + \mathbf{a}_2$, $\mathbf{a}_1 - \mathbf{a}_2$, $2\mathbf{a}_1 + 3\mathbf{a}_2$, $3\mathbf{a}_1 - 5\mathbf{a}_2$, etc. Such vectors are called ***linear combination of vectors*** $\mathbf{a}_1$ and $\mathbf{a}_2$. In general, the vector $\mathbf{b} = \lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. Figure 2·3 explains this point.

Thus, if $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$ is a set of k vectors in n-space and $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ is a set of k scalars, then the vector

$$\mathbf{b} = \lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2 + \ldots + \lambda_k\mathbf{a}_k$$

is a linear combination of the given set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$.

This concept of linear combination provides us with a definition of the line segment between two vectors, or points.

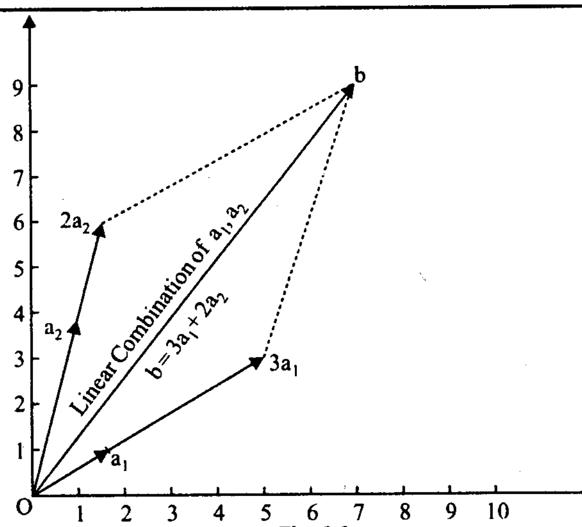


Fig. 2.3
Vector b as Linear Combination of a_1, a_2

Line segment between a_1 and a_2 :

Definition. The line segment between two points (vectors) $\mathbf{a}_1$ and $\mathbf{a}_2$ is the set of points

$$\mathbf{b} = \lambda \mathbf{a}_1 + (1 - \lambda)\, \mathbf{a}_2 \text{, for all } \lambda,\ 0 \le \lambda \le 1 .$$

This definition of the line segment can be extended to any number of points. In its more general form, this linear combination is called a ***convex combination*** and represents a segment of a plane. The plane segment is called the ***convex- hull*** of the points.

Convex–Hull of points $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ in E^n.

Definition. The convex-hull of p points $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ in E^n is the *set of points* $\mathbf{b} = \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \ldots + \lambda_p \mathbf{a}_p$, for all *non-negative* $\lambda_1, \lambda_2, \ldots, \lambda_p$ such that $\lambda_1 + \lambda_2 + \ldots + \lambda_p = 1$.

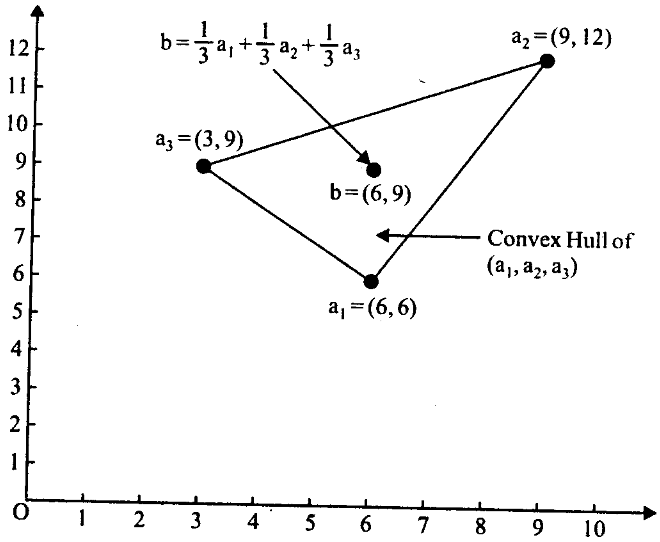


Fig. 2.4
Convex hull of three points in E^2

Example 1. *Consider three points* $a_1 = (6, 6)$, $\mathbf{a}_2 = (9, 12)$, $\mathbf{a}_3 = (3, 9)$.

The convex hull of these three points is shown in the side Fig. 2·4. If $\lambda_1 = \frac{1}{3}, \lambda_2 = \frac{1}{3}, \lambda_3 = \frac{1}{3}$ (so that $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and all $\lambda \ge 0$), then $\mathbf{b} = \frac{1}{3}(6, 6) + \frac{1}{3}(9, 12) + \frac{1}{3}(3, 9) = (6, 9)$.

2.7 LINEAR INDEPENDENCE, SPANNING SET AND BASIS

We now define some useful properties of set of vectors. If, in E^n, we have a set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$, we say that they are ***linearly independent*** if no one of these can be expressed as a linear combinations of the remaining ones.

Linear Independence. A set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ is linearly independent if the equation

$$\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \ldots + \lambda_k \mathbf{a}_k = 0,$$

is satisfied only if $\lambda_1 = \lambda_2 = \ldots = \lambda_k = 0$.

A set of vectors is *linearly dependent* if it is not linearly independent.

Example 1. *The vectors* $\mathbf{a}_1 = (1, 2)$ and $\mathbf{a}_2 = (2, 4)$ *are linearly dependent since there exist* $\lambda_1 = 2$ and $\lambda_2 = -1$ *for which* $\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 = \mathbf{0}$.

Example 2. *The vectors* $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ *and* $e_3 = (0, 0, 1)$ *are linearly independent, since*

$$\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 = \mathbf{0}$$
$$\Rightarrow \lambda_1 (1, 0, 0) + \lambda_2 (0, 1, 0) + \lambda_3 (0, 0, 1) = (0, 0, 0)$$
$$\Rightarrow (\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0)$$
$$\Rightarrow \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0.$$

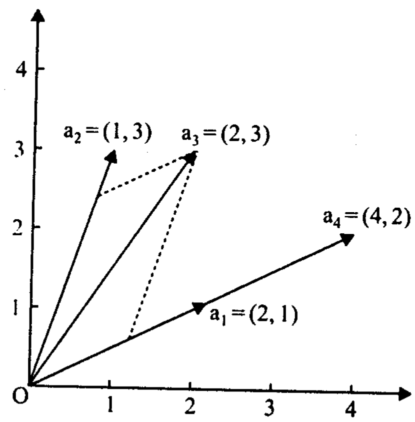


Fig. 2.5
Linear dependence and independence

Example 3. *Consider the set of four vectors* $\mathbf{a}_1 = (2, 1)$, $\mathbf{a}_2 = (1, 3)$, $\mathbf{a}_3 = (2, 3)$, $\mathbf{a}_4 = (4, 2)$. *It can be geometrically seen below that—*

(*i*) The set $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent, since neither of the vectors can be expressed in terms of the others.

(*ii*) The set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is not linearly independent, since $\mathbf{a}_3$ can be expressed as the linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

(iii) The set $\{\mathbf{a}_1, \mathbf{a}_4\}$ is also linearly dependent, since $2\mathbf{a}_1 = \mathbf{a}_4$.

(iv) The set of single vector $\{\mathbf{a}_1\}$ is also linearly independent.

Remark. If a set of vectors is linearly independent, then any subset of it is also linearly independent. If a set of vectors is linearly dependent, then a superset of it is also linearly dependent.

Spanning set. The set of vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ in E^n is a *spanning set* in E^n if every vector in E^n can be expressed as a linear combination of vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$.

From the Fig. 2.5, it can be seen that no less than two vectors are required to form the spanning set in E^2. So, for example, the set $\{\mathbf{a}_1, \mathbf{a}_2\}$ is a spanning set and so is $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. However, the set $\{\mathbf{a}_1\}$ is not a spanning set and neither is $\{\mathbf{a}_1, \mathbf{a}_4\}$ since many vectors in the space cannot be expressed as linear combination of either of these sets.

Basis set. A set of vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ in E^n is a ***basis set*** if—

(i) *it is a linearly independent set, and* (ii) *it is a spanning set of* E^n.

If it is a basis then $k = n$.

Standard basis. The set of unit vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is called the standard basis for E^n (since the set of vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is a linearly independent set, and it is also a spanning set of E^n).

Remark. It will be seen later that the standard basis is the same as the identity matrix which will be the basis for the development of simplex method of linear programming. This may be renamed as basis matrix also.

II–Matrices and Determinants

2.8 MATRICES

Matrix algebra is extremely useful is solving a set of linear equations. Any linear programming problem can be solved with the help of matrix algebra. In particular, the algorithm (a systematic procedure) of the simplex method is based on the concepts of matrices and inversion of matrices. It is also useful in the study of games and optimum strategies. Therefore it is important for the reader to become familiar with the basic properties of this area of mathematics.

2.9 DEFINITIONS

Matrix. *A matrix is a rectangular array of ordered numbers, arranged into rows and columns.*

Given below are several examples of matrices :

Example 1	*Example 2*	*Example 3*
$\begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix}$	$\begin{pmatrix} 1 & 4 & 3 \\ 0 & 2 & 1 \end{pmatrix}$	$\begin{bmatrix} 2 & 1 & 3 \\ 3 & 5 & 4 \\ 4 & 6 & 7 \end{bmatrix}$
2×2 matrix.	2×3 matrix.	3×3 matrix.

Matrix A of size $(m \times n)$ is a rectangular array (table) of ordered numbers arranged into m rows and n columns represented by

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n}$$

Note that a matrix is simply used to convey information in a concise manner and in an acceptable form for mathematical modelling. Taken as a whole, a matrix has no numerical value. The elements a_{ij} for $i = j$, *i.e.* $a_{11}, a_{22}, a_{33}, \dots$, and so on, are called ***principal diagonal elements***, others are called ***off-diagonal elements***.

Square matrix. Any matrix in which the number of rows equals the number of columns (*i.e.*, $m = n$) is called a square matrix.

Example 1 & *Example 3* above are square matrices.

Diagonal matrix. A square matrix in which all off-diagonal elements are zero (*i.e.*, $a_{ij} = 0$ for $i \neq j$) is called a *diagonal matrix.*

For example, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. is a diagonal matrix.

Identity (or Unit) matrix. A diagonal matrix whose all principal diagonal elements are 1 is called an Identity (or Unit) matrix denoted simply by I.

For example, $$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is an identity matrix of third order, sometimes denoted by I_3.

Note. I is necessarily a square matrix.

Transpose matrix. The transpose of a matrix $A = [a_{ij}]$, denoted by A' or A^T is a matrix obtained by interchanging the rows and columns of A. For example,

Original Matrix A $\Rightarrow$ Transpose Matrix A^T

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Symmetric matrix. A square matrix A is said to be symmetric if the matrix A remains the same by interchanging the rows and columns of A (*i.e.*, $a_{ij} = a_{ji}$ or $A^T = A$).

For example, $$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}.$$

Here it is observed that the elements at equal distances above and below the principal diagonal are equal.

Row matrix. A matrix having only a single row is called a *row matrix* (or a *row vector*). It is an $1 \times n$ matrix.

Column matrix. A matrix having only a single column is called a *column matrix* (or a *column vector*). It is an $m \times 1$ matrix.

For example, Row Matrix Column Matrix

$$(1 \;\; 3 \;\; 2) \qquad \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

Null matrix. A matrix whose all elements are zero is called a *null matrix*.

2.10 MATRIX OPERATIONS

Addition and Subtraction. Two matrices can be added only if they have the same dimensions (order). If the number of rows and columns of the two matrices are same, then the sum of two such matrices can be obtained by adding together the respective elements. Matrix addition is also known as *elementwise* addition.

Rules for matrix subtraction are the same as those of matrix addition. The matrix subtraction is an elementwise subtraction. The following are the examples of matrix addition and subtraction.

Example of Addition

Matrix A + Matrix B = Matrix C

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 1 & 6 \end{pmatrix} + \begin{pmatrix} -6 & 2 & 6 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 2-6 & 4+2 & -2+6 \\ 4+1 & 1-1 & 6+2 \end{pmatrix} = \begin{pmatrix} -4 & 6 & 4 \\ 5 & 0 & 8 \end{pmatrix}$$

Example of Subtraction

Matrix A – Matrix B = Matrix C

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 1 & 6 \end{pmatrix} - \begin{pmatrix} -6 & 2 & 6 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 2-(-6) & 4-2 & -2-6 \\ 4-1 & 1-(-1) & 6-2 \end{pmatrix} = \begin{pmatrix} 8 & 2 & -8 \\ 3 & 2 & 4 \end{pmatrix}.$$

Multiplication by a scalar. The *scalar multiplication* of a matrix $A = (a_{ij})$ by a scalar (real number) α is a matrix obtained by multiplying all elements of A by α. For example,

Scalar $3 \times$ Matrix (A) = Matrix $(3A)$

$$3 \times \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 9 & 12 \\ 3 & 6 & 15 \end{pmatrix}.$$

Negative of a matrix. The negative of a matrix $A = (a_{ij})$ is obtained by multiplying all elements of A by scalar -1. For example,

$$-\begin{pmatrix} 2 & 3 & 4 \\ -1 & 2 & 5 \end{pmatrix} = \begin{pmatrix} -2 & -3 & -4 \\ 1 & -2 & -5 \end{pmatrix}$$

Laws governing the addition and scalar multiplication. For any three matrices A, B, C and scalars α, β, the following properties hold :

(i) $A + (B + C) = (A + B) + C$ *(ii)* $A + O = O + A = A$ *(iii)* $A + (-A) = 0 = (-A) + A$
(iv) $A + B = B + A$ *(v)* $\alpha(A + B) = \alpha A + \alpha B$ *(vi)* $(\alpha + \beta) A = \alpha A + \beta A$
(vii) $\alpha(\beta A) = (\alpha\beta) A = \beta(\alpha A)$ *(viii)* $0A = 0$.

Matrix multiplication. Two matrices can be multiplied together if the number of columns in the *first* matrix is equal to the number of rows in the second matrix. Remember, if this condition is not satisfied, the multiplication is impossible.

The following are some examples of matrices that can or cannot be multiplied; consider first matrix (A) and second matrix (B).

First Example :

$$\underset{(A)}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3}} \times \underset{(B)}{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_{3\times 1}}$$

Here number of columns in (A) *is equal* to the number of rows in (B).

So matrix (A) can be multiplied by matrix (B).

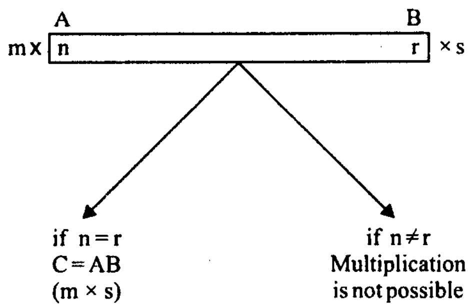


Fig. 2.6

Second Example :

$$\underset{(A)}{\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}_{3\times 1}} \times \underset{(B)}{\begin{pmatrix} 3 & 2 & 1 \\ 5 & 4 & 6 \end{pmatrix}_{2\times 3}}$$

Here number of columns in (A) *is not equal* to the number of rows in (B).

So matrix (A) cannot be multiplied by matrix (B).

Existence of matrix multiplication can also be explained by Fig. 2.6.

If the multiplication is justified, product AB can be computed through row-by-column multiplication rule, as shown below.

Matrix A × Matrix B = Matrix C

$$\begin{matrix} R_1 \rightarrow \\ R_2 \rightarrow \end{matrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3} \times \overset{C_1 \downarrow \;\; C_2 \downarrow}{\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}_{3\times 2}} = \begin{pmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{pmatrix}_{2\times 2} = \begin{pmatrix} 14 & 20 \\ 32 & 47 \end{pmatrix}_{2\times 2}$$

where $R_1C_1 = (1\ 2\ 3) \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = (1 \times 1 + 2 \times 2 + 3 \times 3) = 14$, $R_1C_2 = (1\ 2\ 3) \times \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = (2 + 6 + 12) = 20$

$R_2C_1 = (4\ 5\ 6) \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = (4 + 10 + 18) = 32$, $R_2C_2 = (4\ 5\ 6) \times \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = (8 + 15 + 24) = 47$.

Laws of Matrix Multiplication. Since the matrix multiplication has been defined in terms of multiplication of numbers, the following properties can be easily verified to hold good :

For any three matrices A, B, C and scalar α,

(i) $(AB)C = A(BC)$ *(ii)* $(A + B)C = AC + BC$ *(iii)* $A(B + C) = AB + AC$
(iv) $AI = A = IA^{*}$ *(v)* $\alpha(AB) = (\alpha A)B = A(\alpha B)$ *(vi)* $(AB)^T = B^T A^T$
(vii) $(A + B)^T = A^T + B^T$

provided that the matrix multiplication is justified in each case.

Remarks :

(i) The product BA may not be necessarily defined even when AB is defined.

(ii) Even if AB and BA both are defined it may be possible that $AB \neq BA$.

(iii) Unlike number system, $AB = O$, even when $A \neq O$, $B \neq O$. For example,

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

2.11 DETERMINANT OF A SQUARE MATRIX

The determinant of a square matrix A, denoted by $|A|$, is a number obtained by certain operations on the elements of A. If A is a (2×2) matrix, then

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

For the determinant of third order,

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

For example, if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then

$$|A| = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 4 \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + 7 \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = (45 - 48) - 4(18 - 24) + 7(12 - 15) = 0.$$

Singular and Non-singular Matrices. A square matrix A is said to be ***singular*** if its determinant $|A|$ is equal to zero. If $|A| \neq 0$, then A is called ***non-singular.***

2.12 COFACTOR

Each element of a squared matrix (that is 2×2 and larger) has associated with it a *cofactor*.

Definition. ***A cofactor can be defined as that element or group of elements that remains when a row and a column have been removed from the matrix with appropriate sign.***

In order to determine the sign of the cofactor, it is necessary to add together the location of the row and column which have been removed. If the total is an even number, the sign of the cofactor is unchanged. The odd numebr means the sign is changed.

Example. Consider the matrix $A = \begin{bmatrix} 4 & 8 & 4 \\ 1 & 2 & 1 \\ 6 & 4 & 9 \end{bmatrix}$

To determine the matrix of cofactors C, we let

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}.$$

Cofactors C_{11}, C_{12}, C_{13}, etc. can be easily computed as follows :

$C_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 1 \\ 4 & 9 \end{vmatrix}$ (by deleting *first* row and *first* column) $= + (18 - 4) = 14$.

$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 6 & 9 \end{vmatrix}$ (by deleting *first* row and *second* column) $= -(9 - 6) = -3$.

$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 2 \\ 6 & 4 \end{vmatrix}$ (by deleting *first* row and *third* column) $= + (4 - 12) = -8$.

Other cofactors C_{21}, C_{22}, C_{23}, C_{31}, C_{32} and C_{33} can also be determined in the like manner.

2.13 ADJOINT OF MATRIX

The adjoint of a matrix is the transpose of the matrix of cofactors. For example,

Original Matrix (A)		Matrix of Cofactors (C)		Adjoint (C^T)
$\begin{pmatrix} 4 & 1 \\ 3 & 5 \end{pmatrix}$	$\Rightarrow$	$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -1 & 4 \end{pmatrix}$	$\Rightarrow$	$\begin{pmatrix} 5 & -1 \\ -3 & 4 \end{pmatrix}$

2.14 INVERSE OF A MATRIX

Definition. *The inverse of a non-singular square matrix A is defined by a non-singular square matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$ (Identity Matrix).*

The inverse matrix (A^{-1}) can be obtained by performing row operations on the original matrix A. The row operations are :

1. One row may be interchanged with another row.
2. A row can be multiplied by a number.
3. One row can be added to or subtracted from another row.
4. A multiple of a row can be added to or subtracted from another row.

Thus, to obtain the inverse of a matrix A, we convert the original matrix into an identity matrix of similar size as A by using row operations.

Inverse Formula. The inverse of non-singular square matrix A can also be obtained by using the formula.

$$A^{-1} = \frac{1}{|A|}(\text{adj. } A), \quad |A| \neq 0.$$

Example. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Since $|A| = -2$, A is non-singular, and hence A^{-1} exists. To compute A^{-1}, start with the following matrix

$$(AI) = \begin{bmatrix} 1 & 1 & : & 1 & 0 \\ 1 & -1 & : & 0 & 1 \end{bmatrix}.$$

Subtract row 1 from row 2 : $\begin{bmatrix} 1 & 1 & : & 1 & 0 \\ 0 & -2 & : & -1 & 1 \end{bmatrix}$.

Divide row 2 by -2 : $\begin{bmatrix} 1 & 1 & : & 1 & 0 \\ 0 & 1 & : & 1/2 & -1/2 \end{bmatrix}$.

Subtract row 2 from row 1 : $\begin{bmatrix} 1 & 0 & : & 1/2 & 1/2 \\ 0 & 1 & : & 1/2 & -1/2 \end{bmatrix}$.

Thus $\quad A^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$.

It can be verified that $AA^{-1} = A^{-1}A = I$.

Alternative. By formula, $A^{-1} = \frac{1}{|A|}(\text{adj. } A). = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$.

Properties of inverse of a matrix :

Let A and B be two non-singular matrices of the same order. Then

$(i)(AB)^{-1} = B^{-1}A^{-1}(ii)(A^T)^{-1} = (A^{-1})^T(iii)\,(A^{-1})^{-1} = A$.

2.15 RANK OF A MATRIX

Before defining the rank of a matrix, it will be better to make the concept of rank clear by considering a numerical example.

Let us consider a matrix, $A = \begin{bmatrix} 1 & -3 & 4 \\ 9 & 1 & 2 \end{bmatrix}$.

Clearly, the matrix A possesses a *largest* minor of 2×2 order which may be obtained by deleting any one column. If at least one 2×2 minor is not equal to zero, the rank of A will be 2. If all 2×2 minors are zero, the rank of A will be less than 2. Here one of the (2×2) minors, say

$$\begin{vmatrix} -3 & 4 \\ 1 & 2 \end{vmatrix} = -6 - 4 \neq 0.$$

Hence rank of A will be 2. In case, all (2×2) minors comes out to be zero and the matrix is not a null matrix, then the rank of matrix will be 1.

The rank of null matrix is always zero and the rank of identity matrix of order n will be n.

Definition. *A positive integer r is said to be the* ***rank*** *of a matrix A (denoted by $\rho(A)$) if,*

(i) matrix A possesses at least one r–rowed minor which is not zero, and

(ii) matrix A does not possess any non–zero (r + 1)–rowed minor.

Equivalent Matrices. Two matrices A and B are said to be *equivalent,* if and only if, $\rho(A) = \rho(B)$, denoted by $A \sim B$.

If the matrix A^* can be obtained from matrix A by suitable row (column) operations, then A^* is also called equivalent to A, *i.e.*, $A^* \sim A$.

Properties of rank of matrices :

The following are the properties of the rank of a matrix :

(i) $A^* \sim A \Rightarrow \rho(A^*) = \rho(A)$, *(ii)* $\rho(A^T) = \rho(A)$ for any matrix A, *(iii)* $\rho(A) = r \Leftrightarrow A \sim \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$

(iv) $\rho(AB) \le \min\{\rho(A), \rho(B)\}$ for matrices *A and B, such that AB* is defined.

2.16 QUADRATIC FORMS

In 2-dimension. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be a column vector in 2–space and a (2×2) matrix A given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

then a function of two variables; denoted by $f(x_1, x_2)$ or $Q(\mathbf{x})$, is called a ***quadratic form*** if,

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = (x_1, x_2)\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_{11}x_1^2 + (a_{12} + a_{21})\, x_1x_2 + a_{22}x_2^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij}\, x_i x_j .$$

In 3-dimension. To extend the 2-dimensional case to 3-dimension, we let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in 3-space and

a (3×3) matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

then a function of three variables, denoted by $f(x_1, x_2, x_3)$ or $Q(\mathbf{x})$, is called a quadratic form if

$$Q(\mathbf{x}) = (x_1, x_2, x_3)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + (a_{12} + a_{21})\, x_1x_2 + (a_{13} + a_{31})\, x_1x_3 + (a_{23} + a_{32})\, x_3x_2 ,$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij}\, x_i x_j .$$

In general, we can similarly extend two and three variables quadratic forms to n variables case as follows :

Definition. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $(n \times n)$ matrix $A = [a_{ij}]$, then a function of n variables denoted by $f(x_1, x_2, \ldots, x_n)$ or $Q(\mathbf{x})$, is called a ***quadratic forms in n–sapce***, if

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i x_j .$$

Without any loss of generality, the matrix A can always be assumed as a *symmetric* matrix $(A = A^T)$, because the coefficient of $(x_i x_j)$ is $(a_{ij} + a_{ji})$ (as already seen for *two* and *three* dimensions). In case, matrix A is not symmetric, we can always construct a symmetric matrix B by using the property

$$\mathbf{x}^T B\mathbf{x} = \mathbf{x}^T A\,\mathbf{x} \Rightarrow B = \tfrac{1}{2}(A + A^T), \text{ where } b_{ij} = b_{ji} = \tfrac{1}{2}(a_{ij} + a_{ji}) .$$

Thus B will be symmetric by construction. This assumption has several advantages and hence is taken as a restriction.

Example 1. *Consider the quadratic form*

$$Q(\mathbf{x}) = (x_1, x_2)\begin{bmatrix} 1 & 0 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}^{\mathbf{T}} A\,\mathbf{x}$$

Simplifying we get

$$Q(\mathbf{x}) = 1.x_1^2 + (0+2)\,x_1x_2 + 7x_2^2 = x_1^2 + 2x_1x_2 + 7x_2^2. \qquad \text{...(i)}$$

Now construct the symmetric matrix B by using the property :

$$\therefore \quad \frac{1}{2}(A + A^T) = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 2 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 0 & 7 \end{pmatrix}\right] \quad \text{where } A = \begin{bmatrix} 1 & 0 \\ 2 & 7 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 \\ 2 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 7 \end{bmatrix},$$

which is clearly a symmetric matrix. Now consider the quadratic form

$$Q(\mathbf{x}) = \mathbf{x}^{\mathbf{T}} B\,\mathbf{x} = (x_1, x_2)\begin{bmatrix} 1 & 1 \\ 1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1.x_1^2 + (1+1)\,x_1x_2 + 7x_2^2 = x_1^2 + 2x_1x_2 + 7x_2^2 \qquad \text{...(ii)}$$

From (i) and (ii) we observe that both the quadratic forms, $Q(\mathbf{x})$, are the same.

Example 2. *Verify that the quadratic form*

$$Q(\mathbf{x}) = (x_1, x_2, x_3)\begin{bmatrix} 1 & 0 & 1 \\ 2 & 7 & 6 \\ 3 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x}^{\mathbf{T}} A\mathbf{x}$$

is also same as the quadratic form

$$Q(\mathbf{x}) = (x_1, x_2, x_3)\begin{bmatrix} 1 & 1 & 2 \\ 1 & 7 & 3 \\ 2 & 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x}^{\mathbf{T}} B\,\mathbf{x}$$

It should be verified that $B = \frac{1}{2}(A + A^T)$ is a symmetric matrix in the second case.

Now we are interested in the properties of quadratic forms because sufficiency condition for extreme points may be expressed in terms of a quadratic forms. The following definitions are useful in the investigation of critical points.

Properties of Quadratic Forms :

(i) Positive definite. A quadratic form $Q(\mathbf{x})$ is *positive definite* if and only if $Q(\mathbf{x})$ is positive (> 0) for all $\mathbf{x} \neq \mathbf{0}$.

For example, $\quad Q(\mathbf{x}) = (x_1, x_2)\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2x_1^2 + 4x_2^2$

which is greater than zero (> 0) when $\mathbf{x} \neq \mathbf{0}$. So $Q(x)$ is positive definite.

(ii) Positive semi-definite. A quadratic form $Q(\mathbf{x})$ is positive semi-definite, if and only if, $Q(\mathbf{x})$ is non-negative (≥ 0) for all $\mathbf{x}$ and there exists an $\mathbf{x} \neq \mathbf{0}$ for which $Q(\mathbf{x}) = \mathbf{0}$.

For example, $\quad Q(\mathbf{x}) = (x_1, x_2)\begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 8x_1^2 - 8x_1x_2 + 2x_2^2 = 2(2x_1 - x_2)^2,$

which is zero for all $\mathbf{x}$ satisfying $2x_1 = x_2$, positive for all other $\mathbf{x}$, and thus $Q(\mathbf{x})$ is positive definite.

(iii) Negative definite. A quadratic form $Q(\mathbf{x})$ is negative definite if and only if, $-Q(\mathbf{x})$ is *positive definite.* In other words, $Q(\mathbf{x})$ is negative definite when $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$.

For example, $\quad Q(\mathbf{x}) = (x_1, x_2)\begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2x_1^2 + 2x_1x_2 - 3x_2^2 = -[2x_1^2 - 2x_1x_2 + 3x_2^2]$

which is negative when $\mathbf{x} \neq \mathbf{0}$.

(iv) Negative semi-definite. A quadratic form $Q(\mathbf{x})$ is *negative semi-definite.*, if and only if, $-Q(\mathbf{x})$ is positive semi-definite.

For example, $Q(\mathbf{x}) = (x_1, x_2)\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -x_1^2 + 2x_1x_2 - x_2^2 = -(x_1 - x_2)^2,$

which is less than or equal to zero for all x, and zero for $\mathbf{x} \neq \mathbf{0}$ when $x_1 = x_2$.

(v) Indefinite. A quadratic form $Q(\mathbf{x})$ is *indefinite* if $Q(\mathbf{x})$ is positive for some $\mathbf{x}$ and negative for some other $\mathbf{x}$.

For example, $Q(\mathbf{x}) = (x_1, x_2)\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 - 2x_2^2,$

which is positive for $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ where $x_1 \neq 0$ is any arbitrary number ; and negative for $\mathbf{x} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$ where $x_2 \neq 0$ is any arbitrary real number.

III-Linear Simultaneous Equations

2.17 SIMULTANEOUS EQUATIONS : NATURE OF SOLUTIONS IN DIFFERENT CASES

Let us consider the problem of finding values of variables x_1 and x_2 satisfying simultaneously the linear equations :

$$x_1 + 2x_2 = 8 \text{ and } 3x_1 + x_2 = 9 \quad \ldots(2\cdot1)$$

Definitions. A pair of values for x_1 and x_2 satisfying (2·1) is called a *solution* to the equations. A unique solution exists when only one pair (x_1, x_2) will satisfy the equations and there is *no solution* if no pair of values satisfies the equations.

First we see the problem (2·1) graphically. We plot both the equations in the graph as given in Fig. 2.7.

The problem demands for those values of x_1 and x_2 that are common to both the equations (lines) : $x_1 + 2x_2 = 8$ and $3x_1 + x_2 = 9$. In this case, the answer is a single point $A = (2, 3)$. The solution values are $x_1 = 2$ and $x_2 = 3$.

Next consider another set of equations :

$x_1 + 2x_2 = 8,\ 3x_1 + x_2 = 9$ and $x_1 + x_2 = 4.$ ...(2·2)

As seen in Fig. 2·7, no pair (x_1, x_2) exists that satisfies all the three equations simultaneously. So there is no solution to (2·2). Such simultaneous equations are called ***inconsistent.***

Now changing the last equation of (2·2) by $x_1 + x_2 = 5$, the system becomes :

$x_1 + 2x_2 = 8,\ 3x_1 + x_2 = 9,\ x_1 + x_2 = 5$ . ...(2·3)

In this case, any two of the three equations would have provided the unique solution point A, so one of the equations is said to be ***redundant (in excess).***

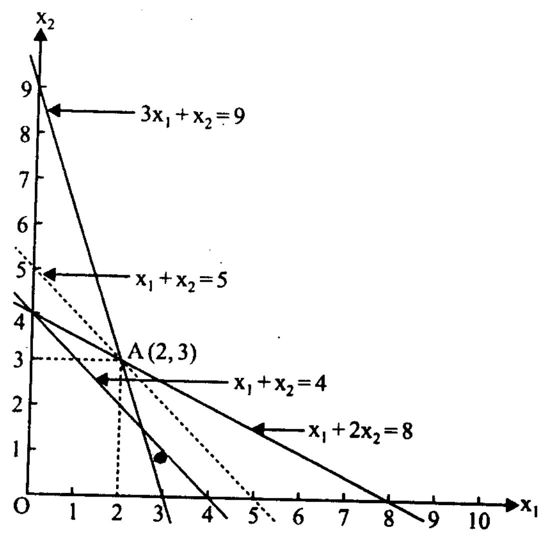


Fig. 2.7
Simultaneous Equations

Again, let us change the problem. We consider the single equation : $x_1 + 2x_2 = 8$ ...(2·4)

Of course, there exist values of x_1 and x_2 satisfying the equation (2·4). But, besides the solution $x_1 = 2, x_2 = 3$ all points on the line $x_1 + 2x_2 = 8$ satisfy this equation. So the number of solutions is infinite. Although, some specific solutions can be obtained by setting one of the variables equal to some value. For example, set $x_2 = 0$, then find $x_1 = 8$. We will find some such solutions particularly useful in linear programming (called *basic solutions* of linear programming problem.)

Summary of results for two-variable case :

We may summarize the results obtained for the simultaneous equations of two variables :

(i) Two variables and more than two equations normally have no solution.

(ii) Two variables and two equations normally have a unique solution.

(iii) Two variables and less than two equations normally have an infinite number of solutions.

(iv) Redundant equations may reduce case (2·1) to case (2·2) or (2·3), and may reduce the case (2·2) to case (2·3).

(v) Inconsistent equations in case (2·2) have no solution.

The results (iv) and (v) generalize to the n-variable case as follows :

(a) n–variables and more than n equations have no solution, unless there are redundant equations (in which case the solution may be unique, or there may be an infinite number of solutions).

(b) n variables and n equations have a unique solution, unless the equations are inconsistent (there is no solution) or unless some equations are redundant (the number of solutions is infinite.)

(c) n variables and less than n equations have an infinite number of solutions, unless there are inconsistent equations (in which case there is no solution).

2.18 NUMERICAL SOLUTION OF SIMULTANEOUS EQUATIONS

For simplicity, again consider the system (2·1) of simultaneous equations :

$$x_1 + 2x_2 = 8$$
$$3x_1 + x_2 = 9.$$

In matrix notation, we can write as

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix} \quad \text{or} \quad \mathbf{Ax} = \mathbf{b}, \qquad \text{...(2·5)}$$

where $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{b} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$.

Now using the fact that $A^{-1}A = I$ and $I\mathbf{x} = \mathbf{x}$, we premultiply both sides of (2·5) by A^{-1} and simplify :

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \text{ or } \mathbf{x} = A^{-1}\mathbf{b}.$$

Thus, formally, the solution to (2·5) can be obtained if we know the inverse of matrix A.

However, this is not an efficient method of solving the equations because, large number of operations are required to find the inverse of A. So the method based on the technique of *Gaussian elimination* require fewer operations.

2.18–1. Gauss-Jordan Method

The solution of simultaneous linear equations can be easily obtained by simple version of Gauss-Jordan method. The step by step procedure is outlined below :

Step 1. The first step is to form the augmented matrix $[A : \mathbf{b}]$ in which the column vector $\mathbf{b}$ now forms the additional column.

Step 2. Then we apply either the following row operations systematically, to obtain a matrix with ones and zeros in appropriate positions :

(i) multiply or divide all elements of a row by some suitable number; or

(ii) replace a row by the sum of that row and a multiple of some other row.

Step 3. Finally, the solution to the original system may be read off.

Following numerical examples will make the procedure clear.

Example 1. *Obtain the solution of the system of simultaneous equations :* $x_1 + 2x_2 = 8$, $3x_1 + x_2 = 9$, *by Gauss-Jordan method.*

Solution :

Step 1. The augmented matrix is obtained as follows : $[A : \mathbf{b}] = \begin{bmatrix} 1 & 2 & : & 8 \\ 3 & 1 & : & 9 \end{bmatrix}$

Step 2.

(i) Subtract 3-times of first row from second row to get, $\begin{bmatrix} 1 & 2 & : & 8 \\ 0 & -5 & : & -15 \end{bmatrix}$

(ii) Divide third row by – 5 to get, $\begin{bmatrix} 1 & 2 & : & 8 \\ 0 & 1 & : & 3 \end{bmatrix}$

(iii) Subtract 2–times of second row from first row to get, $\begin{bmatrix} 1 & 0 & : & 2 \\ 0 & 1 & : & 3 \end{bmatrix}$

Step 3. Rewriting the set of equations, we have $\begin{Bmatrix} x_1 + 0x_2 = 2 \\ 0x_1 + x_2 = 3 \end{Bmatrix}$ which immediately give the solution $x_1 = 2, x_2 = 3$.

Example 2. *Solve the system of equations :*

$$\begin{aligned} x_1 + 2x_2 + \quad x_4 \quad &= 7, \\ x_2 + x_3 + x_4 + x_5 &= 24, \\ x_1 \quad - 3x_3 \quad + 2x_5 &= 8, \end{aligned}$$

for x_1 , x_3 *and* x_5 *in terms of the remaining variables* x_2 *and* x_4 .

Solution. The given system of equations can be written in the matrix form as :

$$\begin{array}{c} \\ \\ R_1 \rightarrow \\ R_2 \rightarrow \\ R_3 \rightarrow \end{array} \begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 0 & 1 & 0 & \vdots\ 7 \\ 0 & 1 & 1 & 1 & 1 & \vdots\ 24 \\ 1 & 0 & -3 & 0 & 2 & \vdots\ 8 \end{array}$$

Since the columns corresponding to variables x_1 , x_3 , and x_5 are indicated by C_1 , C_3 and C_5 respectively, we apply operation *(i)* or *(ii)* to rows R_1 , R_2 and R_3 till the columns C_1 , C_3 and C_5 becomes

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{and} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{respectively.}$$

Thus applying operation $R_3 - R_1$, we get $\begin{vmatrix} 1 & 2 & 0 & 1 & 0 & : & 7 \\ 0 & 1 & 1 & 1 & 1 & : & 24 \\ 0 & -2 & -3 & -1 & 2 & : & 1 \end{vmatrix}$

Applying $3R_2 + R_3$, we get $\begin{vmatrix} 1 & 2 & 0 & 1 & 0 & : & 7 \\ 0 & 1 & 1 & 1 & 1 & : & 24 \\ 0 & 1 & 0 & 2 & 5 & : & 73 \end{vmatrix}$

Applying $1/5\ R_3$, we get $\begin{vmatrix} 1 & 2 & 0 & 1 & 0 & : & 7 \\ 0 & 1 & 1 & 1 & 1 & : & 24 \\ 0 & 1/5 & 0 & 2/5 & 1 & : & 73/5 \end{vmatrix}$

Applying $R_2 - R_3$, we get $\begin{vmatrix} 1 & 2 & 0 & 1 & 0 & : & 7 \\ 0 & 4/5 & 1 & 3/5 & 0 & : & 47/5 \\ 0 & 1/5 & 0 & 2/5 & 1 & : & 73/5 \end{vmatrix}$

Rewriting as a set of equations, we have

$$\begin{aligned} x_1 + 2x_2 \quad + x_4 \quad &= 7 \\ \tfrac{4}{5}x_2 + x_3 + \tfrac{3}{5}x_4 \quad &= 47/5, \\ \tfrac{1}{5}x_2 \quad + \tfrac{2}{5}x_4 + x_5 &= 73/5, \end{aligned}$$

which gives us the required solution :

$$\begin{aligned} x_1 &= 7 - 2x_2 - x_4, \\ x_3 &= \frac{47}{5} - \frac{4}{5}x_2 - \frac{3}{5}x_4, \\ x_5 &= \frac{73}{5} - \frac{1}{5}x_2 - \frac{2}{5}x_4. \end{aligned}$$

2.18–2. Computer Program for Gauss–Jordan Method

A FORTRAN subroutine for solving a system of equations by the Gauss-Jordan method is given below.

```
      SUBROUTINE GJ (A, N)
      DIMENSION A (20, 21)
      N1 = N + 1
      DO 1K = 1 , N
      K1 = K + 1  1
      DO 2J = K1 , N1
2.    A (K , J) = A (K , J)/A (K , K)
      DO 1I = 1 , N
      IF (I . EQ . K) GO TO 1
      DO 3J = K1 , N1
3     A(I , J) = A (I , J) − A(1 , K) * A(K , J)
1     CONTINUE
      RETURN
      END
```

IV–Differentiation and Integration

2.19 INTRODUCTION

Calculus (differentiation and integration) has wide application not only for classical optimization techniques but also for inventory *models, non-linear programming*, and other advanced OR models. So the differentiation and integration are presented along with their formulae for those readers who have elementary knowledge of mathematics.

2.20 DIFFERENTIATION

Differential calculus (differentiation), also called the mathematics of change, is used for determining the slope of a line tangent to a curve at a point on the curve. This concept gives the solution of many complex business problems.

Definition. Let us consider any curve $y = f(x)$; where y is the function of x. If the change in x is Δx and the change in y is Δy, then

$$\Delta y = (y + \Delta y) - y = f(x + \Delta x) - f(x) \qquad \text{and} \qquad \Delta x = (x + \Delta x) - x .$$

The limit of the ratio $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$ is called the *first derivative* of y with respect to x. This is defined by

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

where y is the function of x. The geometric interpretation of the derivative, then, is the slope of a line tangent to a curve at a point.

Higher order differential coefficients are denoted by $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$, etc.

2.20–1. Differentiation Formulae

Mathematicians have developed rules for differentiation with which we can find the derivative of a given differentiable function. The more commonly used formulas where c is a constant, and u and v are the functions of x are listed below.

List of Formulas to Remember :

1. $\frac{d}{dx}(c) = 0$.
2. $\frac{d}{dx}(cx) = c\frac{d}{dx}(x)$
3. $\frac{d}{dx}(x) = 1$
4. $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$
5. $\frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx}$
6. $\frac{d}{dx}(u \,.\, v) = u\frac{dv}{dx} + v\frac{du}{dx}$
7. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
8. $\frac{d}{dx}(u^n) = n\,u^{n-1}\frac{du}{dx}$
9. $\frac{d}{dx}(\log u) = \frac{1}{u}\frac{du}{dx}$
10. $\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$.

2.20–2. Partial Differentiation

Let u be a function of two variables, say x and y, denoted by $u = f(x, y)$. Then by differentiating u partially with respect to x, we mean that y is treated as constant. This is symbolically denoted by $\frac{\delta u}{\delta x}$. Similarly, by $\frac{\delta u}{\delta y}$ we mean the partial derivative of u with respect to y treating x as constant.

Higher order partial derivatives are denoted by $\frac{\delta^2 u}{\delta x^2}, \frac{\delta^2 u}{\delta x\, \delta y}, \frac{\delta^2 y}{\delta x^2}$, etc.

2.21 INTEGRATION

While differentiation is the process of measuring small changes in value, its opposite, the process of summing the intervals under a curve, is called *integration.* A very brief mathematical background is presented here.

Indefinite Integral. The function found with anti-differentiation is called the indefinite integration. The indefinite integral of function f is defined as

$$\int f(x)\,dx = F(x) + C,$$

the constant of integration C must appear in every indefinite integration.

Here, if $f(x)$ is the differentiation of $F(x) + C$, then $F(x) + C$ is called the integration of $f(x)$ with respect to x.

Definite Integral. The definite integral is sometimes called the process of summation. The integral

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a),$$

where the letter a at the bottom of the integral sign is the lower limit of integration and the letter b the upper limit of integration. Since this integral has a definite value equal to the area bounded by the curve $y = f(x)$ and the lines $x = a$ and $x = b$ is called a *definite integral.*

List of Integral Formulae to Remember :

Integration formulas or rules which are part of the mathematical system of calculus are stated below :

1. $\int (0)\,dx = C.$
2. $\int Au\,du = A\int u\,du + C$
3. $\int (1)\,dx = x + C$
4. $\int (u+v)\,dx = \int u\,dx + \int v\,dx + C.$
5. $\int (u-v)\,dx = \int u\,dx - \int v\,dx + C$
6. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C,\ n \neq -1$
7. $\int u^{-1}\,du = \log u + C$
8. $\int e^u\,du = e^u + C$
9. $\int \overset{I}{u}\,.\,\overset{II}{v}\,dx = u\int v\,dx - \int\left(\dfrac{du}{dx} \times \int v\,dx\right)dx$

$$= (I) \times \text{integral of } (II) - \int (\text{Integral of II}) \times \left(\frac{d}{dx}(I)\right)dx$$

2.22 DIFFERENTIATION OF INTEGRALS

We are interested here to find the formula for derivative of the functions given in the form of integrals.

2.22–1 Derivative of Single Integration

Case I. ***Where x depends on z, and f (x, z) has a single definition in the region of integration.***

Let the given function be
$$F(z) = \int_{a(z)}^{b(z)} \overset{II}{f(x, z)}\,\overset{I}{dx}, \qquad \ldots(2\cdot6)$$
where $a(z)$ and $b(z)$ are the limits of integration expressed as the function z.

If $f(x, z)$ possesses a continuous derivative with respect to z throughout the region : $c \le z \le d$, $a(z) \le x \le b(z)$ and, if the derivatives $da(z)/dz$ and $db(z)/dz$ exist, then treating the r.h.s. of (2·6) as the product of two functions, *viz.* $\int_{a(z)}^{b(z)} dx$ as I-function and '$f(x, z)$' as II-function as indicated above, we get the derivative :

$$\frac{dF(z)}{dz} = (\text{I-function}) \times (\text{partial derivative of II w.r.t. 'z'}) + (\text{II-function}) \times (\text{derivative of I w.r.t. 'z'})$$

$$= \int_{a(z)}^{b(z)} \frac{\partial f(x, z)}{\partial z}\,dx + \left[f(x, z)\,.\,\frac{d}{dz}\int_{a(z)}^{b(z)} \frac{dx}{dz}\,dz\right]_{a(z)}^{b(z)}$$

$$= \int_{a(z)}^{b(z)} \frac{\partial f(x, z)}{\partial z}\,dx + \left[f[b(z), z]\,\frac{db(z)}{dz} - f[a(z), z]\,\frac{da(z)}{dz}\right], \qquad \ldots(2\cdot7)$$

for all values of z in the interval $c \le z \le d$. In particular, if $a(z)$ and $b(z)$ are constants, then $da(z)/dz = 0$, $db(z)/dz = 0$. Consequently, we have

$$\frac{dF(z)}{dz} = \int_a^b \frac{\delta f(x, z)}{\partial z}\, dx. \qquad \text{...(2·8)}$$

Furthermore, if the integral on r.h.s. of (2·8) converges, the result is still true when $b \to \infty$, $a \to -\infty$.

Example 3. *Find the derivative of the function* $F(z) = \int_z^{2z} x^2 z^3\, dx$ *(with respect to z).*

Solution. With the help of the result (2·7), we have

$$\frac{dF(z)}{dz} = \frac{d}{dz}\left[\int_z^{2z} \overset{\text{II}}{(x^2}\ \overset{\text{I}}{z^3)}\, dx\right] = \int_z^{2z} \frac{\partial\, (x^2 z^3)}{\partial z}\, dx + \left[x^2 z^3\ \frac{dx}{dz}\right]_z^{2z}$$

$$= \int_z^{2z} 3z^2x^2\, dx + \left[(2z)^2 z^3 \frac{d}{dz}(2z) - z^2 \cdot z^3 \frac{dz}{dz}\right]$$

$$= \int_z^{2z} 3z^2x^2\, dx + 7z^5 = 3z^2 \cdot \left[\frac{x^3}{3}\right]_z^{2z} + 7z^5 = 14z^5 \text{ for any positive value of } z.$$

Case 2. ***When $f(x, z)$ is defined in different forms in the sub-intervals of $0 \le x < \infty$, and x depends on z.***
Particularly, in inventory problems, we often come across to differentiate $F(z)$ defined by

$$F(z) = \int_0^\infty f(x, z)\, dx, \qquad \text{...(2·9)}$$

$$f(x, z) = \begin{cases} f_1(x, z) & \text{for } 0 \le x \le b(z) \\ f_2(x, z) & \text{for } b(z) < x < \infty \end{cases} \qquad \text{...(2·10)}$$

We can find dF/dz only when the function $f(x, z)$ is a continuous function of x, which has a continuous derivative w.r.t. 'z' except possibly at points on the curve $x = b(z)$.

Thus, in view of (2·10), we may express $F(z)$ as,

$$F(z) = \underset{0 \le x \le b(z)}{\int_0^{b(z)} f_1(x, z)\, dx} + \underset{x \ge b(z)}{\int_{b(z)}^\infty f_2(x, z)\, dx} \qquad \text{...(2·11)}$$

Now, using (2·7), we find that

$$\frac{dF}{dz} = \int_0^{b(z)} \frac{\partial f_1}{\partial z}\, dx + \int_{b(z)}^\infty \frac{\partial f_2}{\partial z}\, dx + \frac{db(z)}{dz}\Big[f_1[b(z), z] - f_2[b(z), z]\Big] \qquad \text{...(2·12)}$$

If $f_1[b(z), z] = f_2[b(z), z]$, then obviously dF/dz may be obtained simply by differentiating the expression (2·12) for $F(z)$ under the integration signs. That is,

$$\frac{dF}{dz} = \int_0^{b(z)} \frac{\partial f_1}{\partial z}\, dx + \int_{b(z)}^\infty \frac{\partial f_2}{\partial z}\, dx \qquad \text{...(2·13)}$$

provided $f_1[b(z), z] = f_2[b(z), z]$.

We now proceed to extend the result (2·12) to double integrals by applying it twice, as follows :

2.22–2. Derivative of Double Integration

Case I. ***When $f(x, y, z)$ has a single definition within the limits of integration.***

Let
$$G(z) = \int_{a(z)}^{b(z)} \int_{c(y, z)}^{d(y, z)} f(x, y, z)\, dx\, dy. \qquad \text{...(2·14)}$$

Sufficient conditions for the repeated application of (2·12) are that the derivatives $\partial f/\partial z$, $\partial c/\partial z$, $\partial d/\partial z$ exist and be continuous w.r.t. 'z' throughout the region of integration. Then by writing

$$\int_{c(y, z)}^{d(y, z)} f(x, y, z)\, dx \equiv F(y, z),$$

eqn. (2·14) becomes

$$G(z) = \int_{a(z)}^{b(z)} F(y, z)\, dy.$$

Then, applying (2·12) we have

$$\frac{dG}{dz} = \int_{a(z)}^{b(z)} \frac{\partial F}{\partial z}\, dy + \frac{db(z)}{dz} F[b(z), z] - \frac{da(z)}{dz} F[a(z), z].$$

The result, which is analogous to (2·7), is

$$\frac{dG}{dz} = \int_{a(z)}^{b(z)} \int_{c(y,z)}^{d(y,z)} \frac{\partial f}{\partial z}\, dx\, dy + \int_{a(z)}^{b(z)} \frac{\partial d(y,z)}{\partial z} f[d(y,z), y; z]\, dx - \int_{a(z)}^{b(z)} \frac{\partial c(y,z)}{\partial z} f[c(y,z), y; z]\, dx$$

$$+ \frac{db(z)}{dz} \int_{c[b(z),z]}^{d[b(z),z]} f[x, b(z); z]\, dx - \frac{da(z)}{dz} \int_{c[a(z),z]}^{d[a(z),z]} f[x, a(z); z]\, dx \qquad \text{...(2·15)}$$

Case 2. *When $f(x, y; z)$ is defined in different forms within the region of integration.*

Let a region R of the xy-plane be divided into disjoint subregions R_i where $\Sigma R_i = R$, and suppose that for each sub-region R_i, there corresponds a function $f_i(x, y; z)$. Suppose $F(z)$ is defined by

$$F(z) = \sum_i \int_{R_i} \int f_i(x, y; z)\, dx\, dy.$$

Then, we may calculate dF/dz by $\frac{dF}{dz} = \sum_i \int_{R_i} \int \frac{\partial f_i}{\partial z}\, dx\, dy$,

provided the following two conditions are satisfied.

(*i*) The boundary of the entire region R does not depend on z.

(*ii*) If R_i and R_j are adjacent sub-regions, then $f_i(x, y; z)$ equals $f_j(x, y; z)$ at all points (x, y) along the common boundary of R_i and R_j.

This entire procedure is illustrated by an example in *Model VIII (b)* (of ch. 21 on *Inventory/Production Management-II* on page 159 (Unit 4)).

V– Calculus of Finite Differences

2.23. DIFFERENCE OPERATOR

Let us consider a function $y = f(x)$ defined only for integral values of independent variable x. Here y is a dependent variable.

Suppose we are given equidistant values (finite in number) $a, a+h, a+2h, a+3h, \ldots$ for variable x at an interval of 'h'. Then, the corresponding values of the variable y are :

$$f(a), f(a+h), f(a+2h), \ldots, \text{ and so on.}$$

The values of independent variable x are known as *arguments* and the corresponding values of the dependent variable y are called *entries, e.g.* $f(a+h)$ is the entry corresponding to the argument $a+h$, and so on. The interval 'h' is called the *difference interval.* Unless otherwise stated, the interval of difference is always taken as unity, *i.e.* $h = 1$.

Now we shall define a difference operator Δ which is of fundamental importance in the calculus of finite differences.

Definition. The 'first difference' of $f(x)$ is denoted by $\Delta f(x)$ defined by the formula

$$\Delta f(x) = f(x+1) - f(x) \text{ (taken } h = 1 \text{, here).} \qquad \text{...(2·16a)}$$

Successive differences can also be defined. For example,

$$\Delta^2 f(x) = \Delta[f(x+1) - f(x)] = f(x+2) - 2f(x+1) + f(x). \qquad \text{...(2·16b)}$$

We call Δ^2 the *second-order difference operator* or difference operator of order 2. In general, we define the $(n+1)th$ order difference operator by,

$$\Delta^{n+1} f(x) = \Delta[\Delta^n f(x)], \text{ for } n = 0, 1, 2, 3, \ldots \qquad \text{...(2·17a)}$$

Further, it can be easily verified that

$$\Delta^{n+1} f(x) = \Delta^n [\Delta f(x)] = \Delta^n [f(x+1) - f(x)] = \Delta^n f(x+1) - \Delta^n f(x). \qquad \text{...(2·17b)}$$

Now, we observe that (2·17*b*) is true for all integral values, $n \geq 0$, if

$$\Delta^\circ \equiv \text{Identity operator, e.g. } \Delta^\circ f(x) = f(x) : \text{ and } \Delta^1 \equiv \Delta.$$

2.24 FACTORIAL NOTATION

Definition. The nth factorial of x, denoted by $x^{(n)}$, is defined by

$$x^{(n)} = x(x-1)(x-2)\dots(x-n+1),$$

where by convention $x^{(0)} = 1$.

Similarly, the nth negative factorial of x is defined by $x^{(-n)} = \dfrac{1}{x(x+1)(x+2)\dots(x+n-1)}$.

2.25 FORMULAE FOR FIRST DIFFERENCE OF COMBINATION OF TWO FUNCTIONS

If $f(x)$ and $g(x)$ are any two functions of x, then the following formulae can be readily verified.

1.	$\Delta[f(x)+g(x)] = \Delta f(x) + \Delta g(x)$	6.	$\Delta x^{(n)} = nx^{(n-1)}$, where $x^{(n)}$ is the *factorial function of degree n.*
2.	$\Delta[\alpha f(x)] = \alpha\,\Delta f(x)$, where α is some constant.	7.	In particular, $\Delta x = (x+1) - x = 1$ or $1 = \Delta x$.
3.	$\Delta[f(x)\,g(x)] = f(x)\,\Delta g(x) + g(x+1)\,\Delta f(x)$ $= g(x)\,\Delta f(x) + f(x+1)\,\Delta g(x)$ $= f(x)\,\Delta g(x) + g(x)\,\Delta f(x) + \Delta f(x)\,\Delta g(x)$	8.	$\Delta[a^x] = a^x(a-1)$, where a is some constant.
4.	$\Delta\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)\,\Delta f(x) - f(x)\,\Delta g(x)}{g(x+1)\,g(x)}$	9.	$\Delta[{}^xC_r] = {}^xC_{r-1}$, where r is fixed and ${}^xC_r = \dfrac{x!}{r!\,(x-r)!}$.
5.	$\Delta\left[\dfrac{1}{f(x)}\right] = \dfrac{-\Delta f(x)}{f(x+1)\,f(x)}$		

[**Note.** It is observed that formula (6) is analogous to the formula $D(x^n) = nx^{n-1}$ in differential calculus.

Most of the above formulae are analogous to the corresponding formulae in differential calculus. Hence, these can be remembered easily.]

2.26 CONDITIONS FOR A MINIMUM (MAXIMUM) OF $f(x)$

The function $f(x)$ will have a '*local minimum*' at $x = x_0$, provided both the following conditions are satisfied :

$$f(x_0+1) - f(x_0) > 0,\ i.e.\ \Delta f(x_0) > 0,$$
$$f(x_0) - f(x_0-1) < 0,\ i.e.\ \Delta f(x_0-1) < 0.$$

Thus, we conclude that $f(x)$ will have a 'local minimum' at x_0 if

$$\Delta f(x_0-1) < 0 < \Delta f(x_0). \qquad \dots(2{\cdot}18)$$

The function $f(x)$ is said to have '*absolute minimum*' at x_0 if $f(x_0) \le f(x)$ for all x.

Therefore, sufficient conditions for $f(x)$ to have an '***absolute minimum***' at x_0 are :

$$\Delta f(x_0-1) < 0 < \Delta f(x_0),\ \text{and also that}\ \Delta^2 f(x) \ge 0\ \text{for all}\ x.$$

The sufficient conditions for $f(x)$ to have an ***absolute maximum*** at x_0 are analogous, *i.e.* $\Delta f(x_0-1) > 0 > \Delta f(x_0)$, and also that $\Delta^2 f(x) \le 0$ for all x.

Remark. It is to be noted that these conditions are sufficient for a minimum (maximum), however, they are not necessary. A precise statement of necessary condition is not given here.

2.27 SUMMATION OF SERIES

If $f(x)$ be the continuous function defined for $a \le x \le b$, by fundamental theorem of integral calculus, we know the formula

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a), \qquad \dots(2{\cdot}19)$$

where $F(x)$ is an anti-derivative of $f(x)$.

Now the following theorem will give us an expression quite analogous to (2·14) if $f(x)$ is defined only for the integral values of independent variable x ranging from a to b.

Theorem. *If $f(x)$ be defined only for integral values of independent variable x,* then

$$\sum_{x=a}^{b} f(x) = f(a) + f(a+1) + \dots + f(b) = [F(x)]_a^{b+1} = F(b+1) - F(a),$$

where F(x) is an anti-difference (instead of anti-derivative) of $f(x)$, *i.e.* $\Delta F(x) = f(x)$.

Proof. Since $\Delta F(x) = f(x)$, we have

$$\sum_{x=a}^{b} f(x) = \sum_{x=a}^{b} \Delta F(x) = \sum_{x=a}^{b} [F(x+1) - F(x)]$$

$$= [F(b+1) - F(b)] + [F(b) - F(b-1)] + \dots \dots + [F(a+1) - F(a)]$$

(since $x = b, b-1, b-2, \dots, a+1, a$)

$$= F(b+1) - F(a).$$

Thus, if $F(x)$ is an anti-difference of $f(x)$, *i.e.* $\Delta^{-1} f(x) = F(x)$, then

$$\sum_{x=a}^{b} f(x) = [F(x)]_a^{b+1} = F(b+1) - F(a). \quad \text{...(2·20)}$$

This completes the proof of the theorem.

Note. We note that the upper limit in expression (2·20) is $b+1$ and not b, unlike in expression (2·19).

Example 4. *Use the method of finite differences to find the sum to n terms of the series whose x-th term is given by*

$$f(x) = \frac{x+3}{x(x+1)(x+2)}.$$

Solution. We can express

$$f(x) = (x+3)/x(x+1)(x+2) = [(x+1)(x+2)]^{-1} + 3[x(x+1)(x+2)]^{-1} = x^{(-2)} + 3(x-1)^{(-3)},$$

where $x^{(-r)}$ represents the negative factorial function of x.

Let $\quad F(x) = \Delta^{-1} f(x) = \Delta^{-1}[x^{(-2)} + 3(x-1)^{(-3)}] = \dfrac{x^{(-1)}}{-1} + \dfrac{3(x-1)^{(-2)}}{-2}$

Hence by virtue of result (2·20), we have

$$\sum_{x=1}^{n} f(x) = [F(x)]_1^{n+1} = F(n+1) - F(1)$$

$$= \left[-(n+1)^{(-1)} - \tfrac{3}{2} n^{(-2)}\right] - \left[-1^{(-1)} - \tfrac{3}{2} 0^{(-2)}\right]$$

$$= -\left[\frac{1}{n+2} + \frac{3}{2(n+1)(n+2)}\right] + \left[\frac{1}{2} + \frac{3}{2 \times 1 \times 2}\right] = \frac{5n^2 + 11n}{4(n+1)(n+2)}. \quad \textbf{Ans.}$$

2.28 SUMMATION BY PARTS

As we know from integral calculus, a definite integral of the form $\int_a^b f(x)\, dg(x)$ can be easily evaluated by integrating by parts, as follows :

$$\int_a^b \overset{\text{I}}{f(x)}\, d\overset{\text{II}}{g(x)} = \left[f(x)\, g(x)\right]_a^b - \int_a^b g(x)\, df(x) \quad \text{...(2·21)}$$

$$= [f(b)\, g(b) - f(a)\, g(a)] - \int_a^b g(x)\, df(x),$$

where, $f(x)$ and $g(x)$ are two continuous functions of x defined for $a \le x \le b$.

Now, if $f(x)$ and $g(x)$ are defined only for integral values of x, ranging from a to b, we have a formula nearly analogous to (2·21) in finite differences, which follows :

$$\sum_{x=a}^{b} \overset{\text{I}}{f(x)}\, \Delta \overset{\text{II}}{g(x)} = [f(x)\, g(x)]_a^{b+1} - \sum_{x=a}^{b} g(x+1)\, \Delta f(x) \quad \text{...(2·22)}$$

Example 5. *Evaluate* : $\sum_{x=1}^{k} xa^x$.

Solution. First, we express $\sum_{x=1}^{k} xa^x$ in the form of *l.h.s.* of above formula (2·22). Thus, we have

$$\sum_{x=1}^{k} xa^x = \sum_{x=1}^{k} \overset{\text{I}}{x}\, \Delta \overset{\text{II}}{(a^x/(a-1))} \qquad \text{[because } \Delta a^x = a^x(a-1)\text{]}$$

Hence, applying the above formula (2·22), we have

$$\sum_{x=1}^{k} xa^x = \left[x \cdot \frac{a^x}{a-1} \right]_1^{k+1} - \sum_{x=1}^{k} \left(\frac{a^{x+1}}{a-1} \right) \Delta x$$

$$= \left(\frac{(k+1)\, a^{k+1}}{a-1} - \frac{a}{a-1} \right) - \frac{1}{a-1} \sum_{x=1}^{k} \Delta \left(\frac{a^{x+1}}{a-1} \right) 1 .$$

[because $\Delta a^{x+1} = a^{x+1}(a-1)$ and $\Delta x = 1$].

$$= \frac{(k+1)\, a^{k+1}}{a-1} - \frac{a}{a-1} - \frac{1}{(a-1)^2} \left[a^{x+1} \right]_1^{k+1} \qquad \text{[using the result (2·20)]}$$

$$= \frac{(k+1)\, a^{k+1}}{a-1} - \frac{a}{a-1} - \frac{1}{(a-1)^2} \times [a^{k+2} - a^2] .$$

2.29 DIFFERENCING UNDER SUMMATION SIGN

In Inventory *Model-V* (Chapter ..., Page ...), where the stock level z is a discrete variable, we often feel it necessary to compute the first difference of some function $C(z)$ of the form

$$C(z) = \sum_{x_1, x_2, \dots, x_n} f(x_1, x_2, \dots, x_n ; z) .$$

If $f(x_1, x_2, \dots, x_n ; z)$ has some definition throughout the region of summation, and if the boundary of this region is independent of z, then applying [formula (i) of Sec. 2·25] to each term separately, we obtain

$$\Delta C(z) = \sum_{x_1, x_2, \dots, x_n} \Delta f(x_1, x_2, \dots, x_n ; z) ,$$

where the differences on the right are with respect to z. But, if f is defined in different forms in different parts of the region of summation, and if the boundary of the region depends on z, the computation of $\Delta C(z)$ becomes usually more complicated.

2.29–1 Single Summation

Case 1. ***When x_1 depends on z and $f(x_1, z)$ has a single definition in the region of summation.***

Let us consider the function $C(z) = \sum_{x_1 = a(z)}^{b(z)} f(x_1, z)$.

$$\therefore \quad C(z+1) = \sum_{x_1 = a(z+1)}^{b(z+1)} f(x_1, z+1)$$

$$= \sum_{a(z)}^{b(z)} f(x_1, z+1) + \sum_{b(z)+1}^{b(z+1)} f(x_1, z+1) - \sum_{a(z)}^{a(z+1)-1} f(x_1, z+1)$$

Hence

$$\Delta C(z) = C(z+1) - C(z)$$

$$= \sum_{a(z)}^{b(z)} [f(x_1, z+1) - f(x_1, z)] + \sum_{b(z)+1}^{b(z+1)} f(x_1, z+1) - \sum_{a(z)}^{a(z+1)-1} f(x_1, z+1)$$

$$= \sum_{a(z)}^{b(z)} \Delta f(x_1, z) + \sum_{b(z)+1}^{b(z+1)} f(x_1, z+1) - \sum_{a(z)}^{a(z+1)-1} f(x_1, z+1) \qquad \dots(2·23)$$

Here it is assumed that $a(z)$ and $b(z)$ are the increasing functions of z ; if it is not so, some modification will be required.

Case 2. ***When $f(x_1, z)$ is defined in different forms in different parts of the region of summation.***

We suppose that $f(x_1, z)$ is defined by

$$f(x_1, z) = \begin{cases} f_1(x_1, z), \text{ for } 0 \le x_1 \le b(z) \\ f_2(x_1, z), \text{ for } x_1 > b(z) \end{cases} \quad \text{...(2·24)}$$

and that we wish to compute the difference of the function :

$$C(z) = \sum_{x_1=0}^{\infty} f(x_1, z). \quad \text{...(2·25)}$$

Using (2·22), this can be written as

$$C(z) = \sum_{x_1=0}^{b(z)} f_1(x_1, z) + \sum_{x_1=b(z)+1}^{\infty} f_2(x_1, z).$$

Now, applying the result (2·23) to both the summations, we get

$$\Delta C(z) = \sum_{x_1=0}^{b(z)} \Delta f_1(x_1, z) + \sum_{x_1=b(z)+1}^{\infty} \Delta f_2(x_1, z) + \sum_{b(z)+1}^{b(z+1)} [f_1(x_1, z+1) - f_2(x_1, z+1)] \quad \text{...(2·26)}$$

If $f_1(x_1, z+1) = f_2(x_1, z+1)$ for all x in the interval, $b(z) + 1 \le x \le b(z+1)$, the result (2·26) becomes

$$\Delta C(z) = \sum_{x_1=0}^{\infty} \Delta f(x_1, z). \quad \text{...(2·27)}$$

2.29-2 Double Summation

Case I. ***When x_1, x_2 both depend on z.***

Let us consider the function

$$C(z) = \sum_{x_2=a(z)}^{b(z)} \sum_{x_1=c(x_2, z)}^{d(x_2, z)} f(x_1, x_2; z). \quad \text{...(2·28)}$$

In order to find $\Delta C(z)$, we first define $g(x_2, z)$ by

$$g(x_2, z) = \sum_{x_1=c(x_2, z)}^{d(x_2, z)} f(x_1, x_2; z).$$

Then, (2·28) can be written as $\quad C(z) = \sum_{x_2=a(z)}^{b(z)} g(x_2, z).$

Now, applying the result (2·23), we get

$$\Delta C(z) = \sum_{x_2=a(z)}^{b(z)} \Delta g(x_2, z) + \sum_{b(z)+1}^{b(z+1)} g(x_2, z+1) - \sum_{a(z)}^{a(z+1)-1} g(x_2, z+1). \quad \text{...(2·29)}$$

Again, applying (2·23) to $g(x_2, z)$ we get

$$\Delta g(x_2, z) = \sum_{x_1=c(x_2, z)}^{d(x_2, z)} \Delta f(x_1, x_2; z) + \sum_{d(x_2, z)+1}^{d(x_2, z+1)} f(x_1, x_2; z+1) - \sum_{c(x_2, z)}^{c(x_2, z+1)-1} f(x_1, x_2; z+1)$$

where all differences are w.r.t. 'z'. Hence (2·29) becomes

$$\begin{aligned} \Delta C(z) = & \sum_{x_2=a(z)}^{b(z)} \sum_{x_1=c(x_2, z)}^{d(x_2, z)} \Delta f(x_1, x_2; z) + \sum_{a(z)}^{b(z)} \sum_{d(x_2, z)+1}^{d(x_2, z+1)} f(x_1, x_2; z+1) \\ & - \sum_{a(z)}^{b(z)} \sum_{c(x_2, z)}^{c(x_2, z+1)-1} f(x_1, x_2; z+1) + \sum_{b(z)+1}^{b(z+1)} \sum_{c(x_2, z+1)}^{d(x_2, z+1)} f(x_1, x_2; z+1) \\ & - \sum_{a(z)}^{a(z+1)-1} \sum_{c(x_2, z+1)}^{d(x_2, z+1)} f(x_1, x_2; z+1) \end{aligned} \quad \text{...(2·30)}$$

Hence also, we have assumed that boundary functions a, b, c, d are increasing functions of z.

Case 2. ***When $f(x_1, x_2; z)$ is defined in different forms in different parts of the region of summation.***

Let us suppose that the function $f(x_1, x_2; z)$ be defined (as shown in Fig. (2·8)) as

$$f(x_1, x_2; z) = \begin{cases} f_1(x_1, x_2; z), \text{ for } [0 \le x_1 \le c(x_2, z);\ 0 \le x_2 \le b(z)] \\ f_2(x_1, x_2; z), \text{ for } [c(x_2, z) < x_1;\ 0 \le x_2 \le b(z)] \\ f_3(x_1, x_2; z), \text{ for } [0 \le x_1 \le d(x_2, z);\ b(z) < x_2] \\ f_4(x_1, x_2; z), \text{ for } [d(x_2, z) < x_1;\ b(z) < x_2] \end{cases}$$

Then, $C(z) = \sum_0^\infty \sum_0^\infty f(x_1, x_2; z)$

$$= \sum_0^b \sum_0^c f_1 + \sum_0^b \sum_{c+1}^\infty f_2 + \sum_{b+1}^\infty \sum_0^d f_3 + \sum_{b+1}^\infty \sum_{d+1}^\infty f_4$$

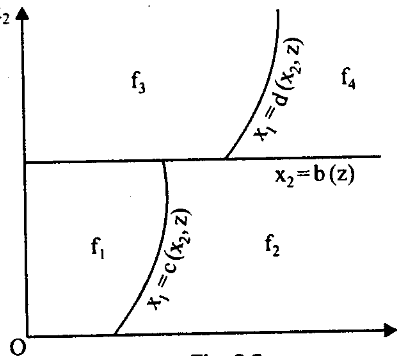


Fig. 2.8
Parts of region of summation in 2-dimension

Now, applying (2·30) to $C(z)$, we get

$$\Delta C(z) = \sum_0^b \sum_0^c \Delta f_1 + \sum_0^b \sum_{c+1}^\infty \Delta f_2 + \sum_{b+1}^\infty \sum_0^d \Delta f_3 + \sum_{b+1}^\infty \sum_{d+1}^\infty \Delta f_4$$

$$+ \sum_0^b \sum_{c+1}^{c(x_2, z+1)} [f_1(x_1, x_2; z+1) - f_2(x_1, x_2; z+1)] + \sum_{b+1}^{b} \sum_{d+1}^{d(x_2, z+1)} [f_3(x_1, x_2; z+1) - f_2(x_1, x_2; z+1)]$$

$$+ \sum_{b+1}^{b(z+1)} \Bigg[\sum_0^{c(x_2, z+1)} f_1(x_1, x_2; z+1) + \sum_{c(x_2, z+1)+1}^\infty f_2(x_1, x_2; z+1) - \sum_0^{d(x_2, z+1)} f_3(x_1, x_2; z+1) - \sum_{d(x_2, z+1)+1}^\infty f_4(x_1, x_2; z+1) \Bigg].$$

The functions f_1, f_2, f_3, f_4 may be of such a form that all terms except the first four cancel out. However, we cannot state sufficient conditions for this to happen as we have done for simple case of single summations. In practical situations, where we have to obtain $\Delta C(z)$ in order to minimize some expected value $C(z)$ and where in the continuous analogue the function being averaged is itself continuous, we suspect that $\Delta C(z)$ in the discrete case will contain only four terms. So we must check such problem to see such simplification is possible.

The example of *Model VIII (a)* (Page 755) will illustrate the procedure for finding $\Delta C(z)$ in that case.

VI–Difference Equations

2.30 DIFFERENCE EQUATION

Definition. *An equation relating the values of a function y and one or more of its differences* Δy, $\Delta^2 y$, ..., *for each value of a set of numbers is called a* ***difference equation.***

The set of numbers to be taken will be non-negative consecutive integers generally starting from zero.

$$\Delta y_k + 4y_k = 0 \quad \text{...(2·31)}$$

$$\Delta^2 y_k + 2\Delta y_k + 3y_k = 0 \quad \text{...(2·32)}$$

$$\Delta y_{k+1} + 4\Delta^3 y_k = 2k + 1 \quad \text{...(2·33)}$$

are examples of difference equations.

Note. All difference operations are taken with a difference interval equal to 1.

Another way of writing a difference equation is as follows :

For example, considering the eqn. (2·31)

$$\Delta y_k + 4y_k = 0 \Rightarrow (y_{k+1} - y_k) + 4y_k = 0 \Rightarrow y_{k+1} + 3y_k = 0 \quad \text{...(2·34)}$$

Similarly, equations (2·32), and (2·33), are equal to

$y_{k+2} + 2y_k = 0$ and $4y_{k+3} - 11y_{k+2} + 11y_{k+1} - 4y_k = 2k + 1$, respectively.

2.30-1 Order of Difference Equation

The difference between the highest and lowest suffix of y [*e.g.*, $k + 1 - k = 1$ in equn. (2·34)] is called the *order of difference equation.*

2.31 LINEAR DIFFERENCE EQUATION WITH CONSTANT COEFFICIENTS

The difference equation

$$y_{k+n} + a_1 y_{k+n-1} + \dots + a_{n-1} y_{k+1} + a_n y_k = f(k), \quad \text{...(2·35)}$$

is called the linear difference equation of order n with constant coefficients if $a_1, a_2, \dots, a_n$ are constants.

If the right hand side of (2·35) is zero, then the equation

$$y_{k+n} + a_1 y_{k+n-1} + \dots + a_{n-1} y_{k+1} + a_n y_k = 0, \quad \text{...(2·36)}$$

is called a ***linear homogeneous equation of order n.***

2.32 SOLUTION OF A DIFFERENCE EQUATION

Consider the difference equation

$y_{k+1} - a y_k = 0$ *(first order)* or $\frac{y_{k+1}}{a^{k+1}} - \frac{y^k}{a_k} = 0$, (dividing by a^{k+1}) or $\Delta\left[y_k/a^k\right] = 0$.

showing that $y^k/a^k = c$, a constant. Therefore, $y_k = ca^k$

Here $y_k = ca^k$ is called the *general solution* of the given *first order* difference equation containing only one arbitrary constant.

But, $y_k = 3a^k$, or $4a^k$ is a particular solution (by giving c a particular value 3 or 4).

If a function is a solution of a difference equation, then it is said to satisfy the equation. The *complete solution* of a general linear equation with constant coefficients given by equation (2·35) is made-up of two parts.

Part 1. ***Complementary function*** (*C.F.*) which is the solution of the homogeneous linear equation given by (2.36) containing n arbitrary constants, equal to the order n of the equation.

Part 2. ***Particular Integral*** (*P.I.*) is a particular solution of (2.35) not containing any arbitrary constant. Then, the complete solution is given by $y_k = \text{C.F} + \text{P.I.}$

2.33 TO FIND THE COMPLEMENTARY FUNCTION (C.F.)

Consider the second order homogeneous linear equation

$$y_{k+2} + a y_{k+1} + b y_k = 0 \quad \text{...(2.37)}$$

Let $y_k = m^k$ be the solution to (2.37). Therefore, substituting the value $y_k = m^k$ in (2.37), we get

$m^{k+2} + am^{k+1} + bm^k = 0$ or $m^k[m^2 + am + b] = 0$ or $m^2 + am + b = 0$, since $m \neq 0$. ...(2.38)

This is called the auxiliary equation (A.E.) of (2.37). Now to solve the auxiliary equation, three different cases may arise.

Case I. ***The roots of A.E. are real and unequal.***

Let m_1 and m_2 be the roots of $m^2 + am + b = 0$. Then, C.F. $= C_1(m_1)^k + C_2(m_2)^k$.

The general solution of the difference equation (2.37) is

$$y_k = C_1(m_1)^k + C_2(m_2)^k. \quad \text{...(2.39)}$$

Case II. ***The roots of the auxiliary equation are real and equal.***

Let both the roots of the auxiliary equation be m. Then, C.F. $= (C_1 + C_2 k)(m)^k$.

The general solution then, will be $y_k = (C_1 + C_2 k)(m)^k$, since $(m)^k$ and $k(m)^k$ both satisfy the equation (2.37). Similarly, if the auxiliary equation has three equal roots, then

$$\text{C.F.} = (C_1 + C_2 k + C_3 k^2)(m)^k. \quad \text{...(2.40)}$$

Case III. ***The roots of the auxiliary equation are imaginary.***

Let $m_1 \pm im_2$ be the roots of the auxiliary equation.

$\therefore \quad$ C.F. $= C_1(m_1 + im_2)^k + C_2(m_1 - im_2)^k$

$= r^k [C_1 (\cos k\theta + i \sin k\theta)] + C_2 (\cos k\theta - i \sin k\theta)$

$\{$where $m_1 = r \cos\theta$, $m_2 = r\sin\theta$, and $(\cos\theta \pm i\sin\theta)^k = \cos k\theta \pm i \sin k\theta\}$

$= r^k [(C_1 + C_2)\cos k\theta + i(C_1 - C_2)\sin k\theta]$ ...(2.41)

We choose the arbitrary constants as

$$C_1 = R[\cos B + i \sin B] \text{ and } C_2 = R[\cos B - i\sin B]$$

$\therefore \quad C_1 + C_2 = 2R\cos B$ and $C_1 - C_2 = 2Ri\sin B$.

Thus, C.F. $= 2Rr^k[\cos k\theta \cos B - \sin k\theta \sin B]$

$= Ar^k \cos(B + k\theta)$, where $A = 2R$, $r = \sqrt{(m_1^2 + m_2^2)}$ and $\theta = \tan^{-1}(m_2/m_1)$.

Illustrative Examples

Example 6. *Solve the equation* $y_{k+2} - 3y_{k+1} + 2y_k = 0$.

Solution. Let the solution of the given equation be $y_k = m^k$.

Then, the auxiliary equation will be

$$m^2 - 3m + 2 = 0 \text{ or } (m-2)(m-1) = 0 \Rightarrow m = 2, 1.$$

$\therefore \quad$ C.F. $= C_1(1)^k + C_2(2)^k$.

Hence the general solution is given by $y_k = C_1 + C_2(2)^k$.

Example 7. *Solve the equation* $y_{k+2} - 4y_{k+1} + 4y_k = 0$.

Solution. Let the solution to the given equation be $y_k = m^k$.

Then, the auxiliary equation will be

$$m^2 - 4m + 4 = 0 \text{ or } (m-2)(m-2) = 0 \Rightarrow m = 2, 2.$$

$\therefore \quad$ C.F. $= (C_1 + C_2 k)\, 2^k$.

Hence the general solution is given by $y_k = (C_1 + C_2 k)\, 2^k$.

Example 8. *Solve the equation* $y_{k+2} - 2y_{k+1} + 4y_k = 0$.

Solution. Let the solution to the given equation be $y_k = m^k$.

Then the auxiliary equation will be

$$m^2 - 2m + 4 = 0 \text{ or } m = \frac{2 \pm \sqrt{(4-16)}}{2} = \frac{2 \pm 2\sqrt{3}\, i}{2} = 1 \pm i\sqrt{3}.$$

If $1 \pm i\sqrt{3} = r(\cos\theta \pm i\sin\theta)$, then

$$r = +\sqrt{(1+3)} = 2, \text{ and } \theta = \tan^{-1}\sqrt{3} = \pi/3.$$

$\therefore \quad$ C.F. $= Ar^k\cos(K\theta + B) = A.2^k \cos\left(K\dfrac{\pi}{3} + B\right)$.

EXAMINATION PROBLEMS

Solve the following equations :

1. $y_{k+2} + 7y_{k+1} + 12y_k = 0$. **[Ans.** $y_k = c_1(-3)^k + c_2(-4)^k$]
2. $y_{k+2} + 4y_{k+1} + 4y_k = 0$. **[Ans.** $y_k = (c_1 + c_2 k)(-2)^k$]
3. $y_{k+3} + 2y_{k+2} - y_{k+1} - 2y_k = 0$. **[Ans.** $c_1 + c_2(-1)^k + c_3(-2)^k$]
4. $y_{k+2} + 9y_k = 0$. $\left[\textbf{Ans. } 3^k\left(c_1\cos\dfrac{k\pi}{2} + c_2\sin\dfrac{k\pi}{2}\right)\right]$
5. $y_{k+2} - 2y_{k+1} + 4y_k = 0$. $\left[\textbf{Ans. } 2^k\left(c_1\cos\dfrac{k\pi}{3} + c_2\sin\dfrac{k\pi}{3}\right)\right]$
6. $(\Delta^2 - 3\Delta + 2)\, y_k = 0$. **[Ans.** $y_k = c_1(2)^k + c_2(3)^k$]

7. $y_{k+2} - y_{k+1} - y_k = 0$, given $y_0 = 0$ and $y_1 = 1$. $\left[\textbf{Ans.}\ \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1-\sqrt{5}}{2}\right)^k\right\}\right]$

8. $y_{k+2} - y_k = 0$, given by $y_0 = 1$ and $y_1 = (1+\sqrt{3})/2$ $\left[\textbf{Ans.}\ y_k = \cos\frac{\pi k}{3} + \sin\frac{k\pi}{3}\right]$

2.34 TO FIND THE PARTICULAR INTEGRAL (P.I.)

Having found out the general solution of the homogeneous equation, we proceed to find out the complete solution of the equation

$$y_{k+n} + a_1 y_{k+n-1} + \ldots + a_{n-1} y_{k+1} + a_n y_k = f(k) \quad \ldots(2.42)$$

or $(E^n + a_1 E^{n-1} + \ldots + a_n E^0)\, y_k = R(k)$, where $E^n y_k = y_{k+n}$, $E^{n-1} y_k = y_{k+n-1}$, etc.

or $f(E)\, y_k = R(k)$, where $f(E) = E^n + a_1 E^{n-1} + \ldots + a_n E^0$.

$$\therefore \quad \text{P.I.} = \frac{R(k)}{f(E)}.$$

Case I. *When* $R(k) = a^k$, P.I. $= a^k/f(E)$.

By the method of undetermined coefficients, suppose $y_k = A\,a^k$ is the particular integral where A is a constant to be determined.

Now $\quad E^n y_k = A\,a^{k+n} = A\,a^k a^n$, $f(E)\, y_k = A\,a^k f(a)$.

Since $y_k = A\,a^k$ is a solution of (2.42), by substitution,

$$A\,a^n f(a) = a^n \Rightarrow \text{or } A = 1/f(a)$$

Thus,
$$\text{P.I.} = A\,a^k = \frac{a^k}{f(a)}$$

$$\therefore \quad \text{P.I.} = \frac{a^k}{f(E)} = \frac{a^k}{f(a)}, \text{ provided } f(a) \neq 0.$$

RULE. *To find the particular integral of* $\frac{a^k}{f(E)}$, *replace E by a, provided* $f(a) \neq 0$.

Failure Case. If $f(a) = 0$, then

$$\text{P.I.} = \frac{a^k}{(E-a)} = k a^{k-1}$$

and
$$\text{P.I.} = \frac{a^k}{(E-a)^2} = \frac{k\,(k-1)}{2\,!} a^{k-2}, \quad \text{and so on.}$$

Illustrative Example

Example 9. *Find the particular integral of*

(i) $2y_{k+2} - 5y_{k+1} + 4y_k = 3^k$, *(ii)* $y_{k+3} - 3y_{k+2} + 4y_k = 2^k$.

Solution. *(i)* The given equation can be written as $(2E^2 - 5E + 4)\, y_k = 3^k$.

$$\therefore \quad \text{P.I.} = \frac{3^k}{2E^2 - 5E + 4} = \frac{3^k}{2(3^2) - 5(3) + 4} = \frac{3^k}{7}. \quad \text{(replacing } E \text{ by 3).}$$

(ii) The given equation can be written as $(E^3 - 3E^2 + 4)\, y_k = 2^k$

$$\therefore \quad \text{P.I.} = \frac{2^k}{E^3 - 3E^2 + 4} = \frac{2^k}{(E+1)(E^2 - 4E + 4)} = \frac{2^k}{(E+1)(E-2)^2} = \frac{2^k}{(2+1)(E-2)^2}$$

$$= \frac{2^k}{3(E-2)^2} = \frac{k(k-1)}{6}(2^{k-2}).$$

Case II. ***When*** $R(k) = \sin ak$ ***or*** $\cos ak$.

Suppose the particular integral in each case be of the form $y_k = A\cos ak + B\sin ak$, where A and B are constants to be determined.

Substituting the value of y_k in the given equation and then equating the coefficients of sin ak and cos ak on both sides of the equation separately, we get two equations in A and B. Thus, A and B can be easily determined.

Special Note. In case this value of y_k fails to give A and B, then suppose $y_k = k\,[A \cos ak + B \sin ak]$

Illustrative Example

Example 10. *Find the particular integral in the equation* $y_{k+2} + y_k = \sin k\pi/2$ ·

Solution. Suppose the particular integral be of the form

$$y_k = k\left[A \cos\frac{k\pi}{2} + B \sin\frac{k\pi}{2}\right],$$

since $y_k = A \cos\frac{k\pi}{2} + B \sin\frac{k\pi}{2}$ fails to give any result. Now substituting the values of y_{k+2} and y_k in the given equation we get,

$$A\,(k+2) \cos (k+2)\frac{\pi}{2} + B\,(k+2) \sin (k+2)\frac{\pi}{2} + Ak \cos\frac{k\pi}{2} + Bk \sin\frac{k\pi}{2} = \sin\frac{k\pi}{2}$$

or
$$-A(k+2) \cos\frac{k\pi}{2} - B\,(k+2) \sin\frac{k\pi}{2} + Ak \cos\frac{k\pi}{2} + Bk \sin\frac{k\pi}{2} = \sin\frac{k\pi}{2}\,.$$

or
$$-2A \cos\frac{k\pi}{2} - 2B \sin\frac{k\pi}{2} = \sin\frac{k\pi}{2}\,.$$

On comparing the coefficients, we get $A = 0$ and $B = -\frac{1}{2}$ ·

$$\therefore \qquad \text{P.I.} = -\frac{k}{2} \sin\frac{k\pi}{2}\,.$$

Case III. ***When*** $R(k) = k^n$.

$$\text{P.I.} = \frac{k^n}{f(E)} = \frac{k^n}{f(1+\Delta)} = [f(1+\Delta)]^{-1}\, k^n\,. \text{ (since } E \equiv 1 + \Delta)$$

Since $\Delta^n\, k^{(m)} = m\,(m-1)\,(m-2) \ldots (m-n+1)\, k^{(m-n)}$, where $k^{(m)}$ is the factorial notation, this formula can be used after expressing k^n in terms of factorial notations.

RULE.	(i)	*Express* k^n *in the factorial notation.*
	(ii)	*Expand* $[f(1+\Delta)]^{-1}$ *by binomial theorem up to the term containing* Δ^n .

Example 11. Solve : $y_{k+2} + 2y_{k+1} + y_k = 3k^2$.

Solution. The r.h.s. can be expressed as $3k^2 = 3k\,(k-1) + 3k = 3k^{(2)} + 3k^{(1)}$. The given equation is

$$(E^2 + 2E + 1)\, y_k = 3k^2\,.$$

The A.E. is $\quad m^2 + 2m + 1 = 0, \quad$ which gives $\quad m = -1, -1$. C.F. $= (c_1 + c_2 k)\,(-1)^k$.

$$\text{P.I.} = \frac{3k^2}{(E+1)^2} = \frac{3k^2}{[2+\Delta]^2} = \frac{3}{4}\left(1 + \frac{\Delta}{2}\right)^{-2} [k^{(2)} + k^{(1)}] \qquad (\because\ E = 1 + \Delta)$$

$$= \frac{3}{4}\left[1 - \Delta + \frac{3}{4}\Delta^2 + \ldots\right][k^{(2)} + k^{(1)}] = \frac{3}{4}\left[k^{(2)} + k - 2k - 1 + \frac{3}{4}(2)\right]$$

$$= \frac{3}{4}\left[k\,(k-1) - k + \frac{1}{2}\right] = \frac{3}{8}\,[2k^2 - 4k + 1]\,.$$

The complete solution is

$$y_k = (c_1 + c_2 k)\,(-1)^k + \frac{3}{8}\,(2k^2 - 4k + 1)\,.$$

Case IV. *Let* $R(k) = a^k k^n$ *(product of* a^k *and* k^n*).*

$$\text{P.I.} = \frac{a^k k^n}{f(E)} = a^k \cdot \frac{1}{f(aE)}\,(k^n) \cdot \text{(replace } E \text{ by } aE)$$

Now proceed as in Case III.

Example 12. *Find the particular integral* of

$$y_{k+3} - 5y_{k+2} + 8y_{k+1} - 4y_k = 2^k \, . \, k^2$$

Solution. The equation can be written as

$$(E^3 - 5E^2 + 8E - 4)\, y_k = 2^k \, . \, k^2 \, .$$

$$\text{P.I.} = \frac{2^k . k^2}{(E^3 - 5E^2 + 8E - 4)} = \frac{2^k \, . \, k^2}{(E-1)(E-2)^2}$$

$$= 2^k \cdot \frac{k^2}{(2E-1)(2E-2)^2}, \quad \text{replacing } E \text{ by } 2E.$$

$$= 2^k \cdot \frac{k^2}{[2(1+\Delta)-1]\,[4\Delta^2]} = 2^k \cdot \frac{k^2}{4\Delta^2\,(1+2\Delta)}$$

$$= \frac{2^k}{4\Delta^2}\,[(1+2\Delta)^{-1}]\,[k^{(2)}+k] = \frac{2^{k-2}}{\Delta^2}\,(1 - 2\Delta + 4\Delta^2)\,[k^{(2)}+k]$$

$$= \frac{2^{k-2}}{\Delta^2}\,[k^{(2)} + k - 4k - 2 + 8] = 2^{k-2} \cdot \frac{1}{\Delta^2}\,[k^{(2)} - 3k + 6]$$

$$= 2^{k-2} \cdot \left[\frac{k^{(4)}}{12} - \frac{k^{(3)}}{2} + 3k^{(2)}\right], \quad \text{since} \left(\frac{k^{(n)}}{\Delta} = \frac{k^{(n+1)}}{n+1}\right).$$

$$= \frac{2^{k-2}}{12}\,[k\,(k-1)(k-2)(k-3) - 6k\,(k-1)(k-2) + 36k\,(k-1)]$$

$$= \frac{2^{k-4}}{3}\,[k^4 - 12k^3 + 65k^2 - 54k]\,.$$

> **Rule.** *For evaluating the particular integral of the product a^k sin bk or a^k cos bk, assume the form of particular integral as*
>
> $$y_k = a^k\,[A\cos bk + B \sin bk]$$

and then proceed as in Case II.

Example 13. *Solve* : $y_{k+1} - 3y_k = 2^k \cos \dfrac{k\pi}{2}$

Solution. The equation can be written as $(E-3)\,y_k = 2^k \cos (k\pi/2)$

The A.E. is $(m-3) = 0$, which gives $m = 3$. ∴ C.F. $= c_1\, 3^k$.

Let the particular integral be $y_k = 2^k \left[A \cos \dfrac{k\pi}{2} + B \sin \dfrac{k\pi}{2}\right]$.

Substituting y_k in the given equation, we have

$$2^{k+1}\left[A\cos(k+1)\frac{\pi}{2} + B \sin (k+1)\frac{\pi}{2}\right] - 3\,.\,2^k\left[A \cos\frac{k\pi}{2} + B \sin \frac{k\pi}{2}\right] = 2^k \cos \frac{k\pi}{2}$$

or $$2\left(-A \sin \frac{k\pi}{2} + B \cos \frac{k\pi}{2}\right) - 3\left(A \cos \frac{k\pi}{2} + b \sin \frac{k\pi}{2}\right) = \cos \frac{k\pi}{2},$$

or $$(-2A - 3B)\sin \frac{k\pi}{2} + (2B - 3A)\cos\frac{k\pi}{2} = \cos \frac{k\pi}{2}.$$

∴ $$-2A - 3B = 0\,, \text{ and } 2B - 3A = 1\,.$$

∴ $$A = -\frac{3}{13}\,;\, B = \frac{2}{13}.$$

The solution is,

$$y_k = c_1\,(3)^k + 2^k\left[-\frac{3}{13}\cos\frac{k\pi}{2} + \frac{2}{13}\sin\frac{k\pi}{2}\right].$$

EXAMINATION PROBLEMS

Solve the following difference equations :

1. $y_{k+2} - 3y_{k+1} - 4y_k = 3^k$.
2. $u_{k+2} - 7u_{k+1} + 10u_k = 12e^{3k} + 4^k$. $\left[\textbf{Hint. } \text{P.I.} = \dfrac{12(e^3)^k + 4^k}{E^2 - 7E + 10}, \text{ replace } E \text{ by } e^3 \text{ and } 4\right]$
3. $y_{k+1} - 4y_k = 2^k$ $\left[\textbf{Hint. } \text{P.I.} = \dfrac{2^k}{E^2 - 4} = \dfrac{2^k}{(E+2)(E-2)} = -\dfrac{k2^{k-1}}{4}\right]$
4. $u_{k+2} - 4u_{k+1} + 4u_k = 2^k$. $\left[\textbf{Hint. } \text{P.I.} = \dfrac{2^k}{(E-2)^2} = \dfrac{k(k-1)}{2!} \cdot 2^{k-2}\right]$
5. $8y_{k+2} - 6y_{k+1} + y_k = 5 \sin k\pi/2$.
6. $y_{k+2} - 2y_{k+1} + y_k = 5 + 3k$.
7. $y_{k+2} - 4y_{k+1} + 4y_k = 3k + 2^k$.
8. $y_{k+2} - 2y_{k+1} + y_k = 2^k \cdot k^2$.

Answers

1. $y_k = c_1(-1)^k + c_2(4)^k - \dfrac{3^k}{4}$.
2. $u_k = c_1(5)^k + c_2(2)^k + 2^{2k-1} + 12\, e^{3k}/(e^6 + 7e^2 + 10)$.
3. $y_k = c_1 2^k + c_2(-2)^k + k2^{k-3}$.
4. $y_k = (c_1 + c_2 k)\, 2^k + k(k-1)\, 2^{k-2}$.
5. $y_k = c_1\left(\dfrac{1}{2}\right)^k + c_2\left(\dfrac{1}{4}\right)^k + \dfrac{1}{17}\left(6 \cos \dfrac{k\pi}{2} - 7 \sin \dfrac{k\pi}{2}\right)$
6. $y_k = c_1 + c_2 k + k^2 + \dfrac{1}{2}k^3$. 7. $y_k = (c_1 + c_2 k)\, 2^k + 6 + 3k + 1/8k^2 \cdot 2k$
8. $y_k = c_1 + c_2 k + 2^k (k^2 - 8k + 20)$.

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UNIT 2

LINEAR PROGRAMMING AND ITS APPLICATIONS

CONTAINING :

Chapter 3. LINEAR PROGRAMMING PROBLEM
(Formulation and Graphical Method)
Chapter 4. CONVEX SETS & ANALYTICAL METHOD
Chapter 5. LINEAR PROGRAMMING : SIMPLEX METHOD
Chapter 6. REVISED SIMPLEX METHOD
I - Revised Simplex Method in Standard Form - I
II - Revised Simplex Method in Standard Form - II
Chapter 7. DUALITY IN LINEAR PROGRAMMING
Chapter 8. THE DUAL SIMPLEX METHOD
Chapter 9. SENSITIVITY ANALYSIS
Chapter 10. INTEGER LINEAR PROGRAMMING
I - Gomory's Cutting Plane Method
II - Banch-and-Bound Method
Chapter 11. TRANSPORTATION PROBLEMS
Chapter 12. ASSIGNMENT PROBLEMS
Chapter 13. MULTI-CRITERIA DECISION PROBLEMS
(Goal Programming)

3

LINEAR PROGRAMMING PROBLEM
(FORMULATION AND GRAPHICAL METHOD)

3.1. INTRODUCTION

In 1947, George Dantzig and his Associates, while working in the U.S. department of Air Force, observed that a large number of military programming and planning problems could be formulated as maximizing/minimizing a linear form of profit/cost function whose variables were restricted to values satisfying a system of linear constraints (a set of linear equations/or inequalities). A linear form is meant a mathematical expression of the type $a_1x_1 + a_2x_2 + \dots + a_nx_n$, where $a_1, a_2, \dots, a_n$ are constants, and $x_1, x_2, \dots, x_n$ are variables. The term '*Programming*' refers to the process of determining a particular programme or plan of action. So Linear Programming (L.P.) is one of the most important optimization (maximization/minimization) techniques developed in the field of Operations Research (O.R.).

The methods applied for solving a linear programming problem are basically simple problems, a solution can be obtained by a set of simultaneous equations. However, a *unique* solution for a set of simultaneous equations in n-variables $(x_1, x_2, \dots, x_n)$, at least one of them is non-zero, can be obtained if there are exactly n relations. When the number of relations is greater than or less than n, a unique solution does not exist, but a number of trial solutions can be found. In various practical situations, the problems are seen in which the number of relations is not equal to the number of variables and many of the relations are in the form of inequalities ($\leq$ or $\geq$) to maximize (or minimize) a linear function of the variables subject to such conditions. Such problems are known as *Linear Programming Problems* (LPP).

In this chapter, properties of LP problems are discussed and at present the graphical method of solving a LPP is applicable where two (or at most three) variables are involved. The most widely used method for solving LP problems of any number of variables is called the *simplex method* developed by G. Dantzig in 1947 and made generally available in 1951.

Definition. *The general LPP calls for optimizing (maximizing/minimizing) a linear function of variables called the 'OBJECTIVE FUNCTION' subject to a set of linear equations and / or inequalities called the 'CONSTRAINTS' or 'RESTRICTIONS'.*

3.2. FORMULATION OF LP PROBLEMS

Now it becomes necessary to present a few interesting examples to explain the real-life situations where LP problems may arise. The outlines of formulation of the LP problems are explained with the help of these examples.

Model Examples on Formulation

Example 1. (Production Allocation Problem) *A firm manufactures two type of products A and B and sells them at a profit of Rs. 2 on type A and Rs. 3 on type B. Each product is processed on two machines G and H. Type A requires one minute of processing time on G and two minutes on H ; type B requires one minute on G and one minute on H. The machine G is available for not more than 6 hour 40 minutes while machine H is available for 10 hours during any working day.*

Formulate the problem as a linear programming problem. **[Kanpur 96]**

Formulation. Let x_1 be the number of products of type A and x_2 the number of products of type B.

After carefully understanding the problem the given information can be systematically arranged in the form of the following table.

Table 3.1

Machine	Time of Products (minutes)		Available Time (minutes)
	Type A (x_1 units)	Type B (x_2 units)	
G	1	1	400
H	2	1	600
Profit per unit	Rs. 2	Rs. 3	

Since the profit on type A is Rs. 2 per product, $2x_1$ will be the profit on selling x_1 units of type A. Similarly, $3x_2$ will be the profit on selling x_2 units of type B. Therefore, total profit on selling x_1 units of A and x_2 units of B is given by

$$\mathrm{P} = 2x_1 + 3x_2 \quad \textit{(objective function)}$$

Since machine G takes 1 minute time on type A and 1 minute time on type B, the total number of minutes required on machine G is given by : $x_1 + x_2$.

Similarly, the total number of minutes required on machine H is given by $2x_1 + x_2$.

But, machine G is not available for more than 6 hour 40 minutes (= 400 minutes). Therefore,

$$x_1 + x_2 \leq 400 \quad \textit{(first constraint)}$$

Also, the machine H is available for 10 hours only, therefore,

$$2x_1 + x_2 \leq 600 \quad \textit{(second constraint)*}$$

Since it is not possible to produce negative quantities,

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \quad \textit{(non-negativity restrictions)}$$

Hence the allocation problem of the firm can be finally put in the form :

Find x_1 and x_2 such that the profit P = $2x_1 + 3x_2$ is maximum,
subject to the conditions :
$x_1 + x_2 \leq 400, \quad 2x_1 + x_2 \leq 600, \quad x_1 \geq 0, \; x_2 \geq 0.$

Example 2. *A company produces two types of Hats. Each hat of the first type requires twice as much labour time as the second type. If all hats are of the second type only, the company can produce a total of 500 hats a day. The market limits daily sales of the first and second type to 150 and 250 hats. Assuming that the profits per hat are Rs. 8 for type A and Rs. 5 for type B, formulate the problem as a linear programming model in order to determine the number of hats to be produced of each type so as to maximize the profit.*

Formulation. Let the company produce x_1 hats of type A and x_2 hats of type B each day. So the profit P after selling these two products is given by the linear function :

$$\mathrm{P} = 8x_1 + 5x_2 \quad \textit{(objective function)}$$

Since the company can produce at the most 500 hats in a day and A type of hats require twice as much time as that of type B, production restriction is given by $2tx_1 + tx_2 \leq 500\,t$, where t is the labour time per unit of second type, *i.e.*

$$2x_1 + x_2 \leq 500 .$$

But, there are limitations on the sale of hats, therefore further restrictions are :

$$x_1 \leq 150, \quad x_2 \leq 250 .$$

Also, since the company cannot produce negative quantities,

$$x_1 \geq 0 \text{ and } x_2 \geq 0 .$$

Hence the problem can be finally put in the form :

Find x_1 and x_2 such that the profit P = $8x_1 + 5x_2$ is maximum,
subject to the restrictions :
$2x_1 + x_2 \leq 500, x_1 \leq 150, x_2 \leq 250, x_1 \geq 0, \; x_2 \geq 0.$

*Here the constraint $2x_1 + x_2 = 600$ is not justified because using machine H for less than 10 hrs (if possible) will be more profitable.

Example 3. *The manufacturer of patent medicines is proposed to prepare a production plan for medicines A and B. There are sufficient ingredient available to make 20,000 bottles of medicine A and 40,000 bottles of medicine B, but there are only 45,000 bottles into which either of the medicines can be filled. Further, it takes three hours to prepare enough material to fill 1000 bottles of medicine A and one hour to prepare enough material to fill 1000 bottles of medicine B, and there are 66 hours available for this operation. The profit is Rs. 8 per bottle for medicine A and Rs. 7 per bottle for medicine B.*

(i) *Formulate this problem as a L.P.P.*

(ii) *How the manufacturer schedule his production in order to maximize profit.*

Formulation. (i) Suppose the manufacturer produces x_1 and x_2 thousand of bottles of medicines A and B, respectively. Since it takes three hours to prepare 1000 bottles of medicine A, the time required to fill x_1 thousand bottles of medicine A will be $3x_1$ hours. Similarly, the time required to prepare x_2 thousand bottles of medicine B will be x_2 hours. Therefore, total time required to prepare x_1 thousand bottles of medicine A and x_2 thousand bottles of medicine B will be $3x_1 + x_2$ hours.

Now since the total time available for this operation is 66 hours, $3x_1 + x_2 \le 66$.

Since there are only 45 thousand bottles available for filling medicines A and B, $x_1 + x_2 \le 45$.

There are sufficient ingredients available to make 20 thousand bottles of medicine A and 40 thousand bottles of medicine B, hence $x_1 \le 20$ and $x_2 \le 40$.

Number of bottles being non-negative, $x_1 \ge 0$, $x_2 \ge 0$.

At the rate of Rs. 8 per bottle for type A medicine and Rs. 7 per bottle for type B medicine, the total profit on x_1 thousand bottles of medicine A and x_2 thousand bottles of medicine B will become

$$P = 8 \times 1000\, x_1 + 7 \times 1000\, x_2 \quad \text{or} \quad P = 8000\, x_1 + 7000\, x_2 .$$

Thus, the linear programming problem is :

Max. $P = 8000\,x_1 + 7000\,x_2$ **, subject to the constraints :**

$$3x_1 + x_2 \le 66, x_1 + x_2 \le 45, x_1 \le 20, x_2 \le 40$$

and $x_1 \ge 0 , x_2 \ge 0$ **.**

(ii) See ***Example* 28** (page 79) for its solution by graphical method.

Example 4. *A toy company manufactures two types of doll, a basic version—doll A and a delux version—doll B. Each doll of type B takes twice as long to produce as one of type A, and the company would have time to make a maximum of 2000 per day. The supply of plastic is sufficient to produce 1500 dolls per day (both A and B combined). The deluxe version requires a fancy dress of which there are only 600 per day available. If the company makes a profit of Rs. 3.00 and Rs. 5.00 per doll, respectively on doll A and B, then how many of each doll should be produced per day in order to maximize the total profit. Formulate this problem.* **[Kanpur B.Sc. 90; Meerut 90]**

Formulation. Let x_1 and x_2 be the number of dolls produced per day of type A and B, respectively. Let the doll A require t hrs so that the doll B require $2t$ hrs. So the total time to manufacture x_1 and x_2 dolls should not exceed 2,000t hrs. Therefore, $tx_1 + 2tx_2 \le 2000t$. Other constraints are simple. Then the linear programming problem becomes :

Maximize $P = 3x_1 + 5x_2$

subject to the restrictions

$$x_1 + 2x_2 \le 2000 \quad \textbf{(time constraint)}$$
$$x_1 + x_2 \le 1500 \quad \textbf{(plastic constraint)}$$
$$x_2 \le 600 \quad \textbf{(dress constraint)}$$

and non–negativity restrictions

$$x_1 \ge 0 , x_2 \ge 0 .$$

Note : See ***Example* 26** (page 76) for its solution by graphical method.

Example 5. *In a chemical industry, two products A and B are made involving two operations. The production of B also results in a by-product C. The product A can be sold at Rs. 3 profit per unit and B at Rs. 8 profit per unit. The by-product C has a profit of Rs. 2 per unit, but it cannot be sold as the destruction cost is Re. 1 per unit. Forecasts show that up to 5 units of C can be sold. The company gets 3 units of C for each unit of A and B produced. Forecasts show that they can sell all the units of A and B produced. The manufacturing times are 3 hours per unit for A on operation one and two respectively and 4 hours and 5 hours per unit for B on operation one and two respectively. Because the product C results from producing B, no time is used in producing C. The available times are 18 and 21 hours of operation one and two respectively. The company question : how much A and B should be produced keeping C in mind to make the highest profit. Formulate LP model for this problem.*

Formulation. Let x_1 , x_2 , x_3 be the number of units produced of product A , B , C respectively. Then the profit gained by the industry is given by $\mathbf{P} = 3x_1 + 8x_2 + 2x_3$.

Here it is assumed that all the units of product A and B are sold.

In first operation, A takes 3 hours of manufacturer's time and B takes 4 hours of manufacturer's time, therefore total number of hours required in first operation becomes $3x_1 + 4x_2$.

In second operation, A takes 3 hours of manufacturer's time and B takes 5 hours of manufacturer's time, therefore the total number of hours used in second operation becomes $3x_1 + 5x_2$.

Since there are 18 hours available in first operation and 21 hours in second operation, the restrictions become : $3x_1 + 4x_2 \le 18$, $3x_1 + 5x_2 \le 21$.

Also, the company gets 3 units of by-product C for each unit of B produced, therefore the total number of units of product B and C produced becomes : $x_2 + 3x_3$.

But, the maximum number of units of C can be sold is 5, therefore $x_2 + 3x_3 \le 5$.

Thus the allocation problem of the industry can be finally put in the form :

Find the value of x_1 , x_2 , x_3 so as to maximize

$\mathbf{P} = 3x_1 + 8x_2 + 2x_3$ subject to the restrictions :

$$3x_1 + 4x_2 \le 18$$
$$3x_1 + 5x_2 \le 21$$
$$x_2 + 3x_3 \le 5,$$

with non-negativity conditions : $x_1 \ge 0$, $x_2 \ge 0$, $x_3 \ge 0$.

Example 6. *A firm can produce three types of cloth, say : A, B, and C. Three kinds of wool are required for it, say : red, green and blue wool. One unit length of type A cloth needs 2 meters of red wool and 3 meters of blue wool ; one unit length of type B cloth needs 3 meters of red wool, 2 meters of green wool and 2 meters of blue wool ; and one unit of type C cloth needs 5 meters of green wool and 4 meters of blue wool. The firm has only a stock of 8 meters of red wool, 10 meters of green wool and 15 meters of blue wool. It is assumed that the income obtained from one unit length of type A cloth is Rs. 3.00, of type B cloth is Rs. 5.00, and of type C cloth is Rs. 4.00.*

Determine, how the firm should use the available material so as to maximize the income from the finished cloth.

Formulation. It is often convenient to construct the *Table 3.2* after understanding the problem carefully.

Table 3.2

Quality of wool	*Type of Cloth*			*Total quantity of wool available (in meters)*
	$A\ (x_1)$	$B\ (x_2)$	$C\ (x_3)$	
Red	2	3	0	8
Green	0	2	5	10
Blue	3	2	4	15
Income per unit length of cloth	Rs. 3.00	Rs. 5.00	Rs. 4.00	

Let x_1 , x_2 and x_3 be the quantity (in meters) produced of cloth type *A* , *B* , *C* respectively. Since 2 meters of red wool are required for each meter of cloth *A* and x_1 meters of this type of cloth are produced, so $2x_1$ meters of red wool will be required for cloth *A*.

Similarly, cloth *B* requires $3x_2$ meters of red wool and cloth *C* does not require red wool. Thus, total quantity of red wool becomes :

$$2x_1 + 3x_2 + 0x_3 \ (\textit{red wool})$$

Following similar arguments for ***green*** and ***blue*** wool,

$$0x_1 + 2x_2 + 5x_3 \ (\textit{green wool})$$
$$3x_1 + 2x_2 + 4x_3 \ (\textit{blue wool})$$

Since not more than 8 meters of red, 10 meters of green and 15 meters of blue wool are available, the variables x_1 , x_2 , x_3 must satisfy the following restrictions :

$$\begin{aligned} 2x_1 + 3x_2 \qquad &\le 8 \\ 2x_2 + 5x_3 &\le 10 \\ 3x_1 + 2x_2 + 4x_3 &\le 15\,. \end{aligned} \qquad \text{...(3.1)}$$

Also, negative quantities cannot be produced. Hence x_1 , x_2 , x_3 must satisfy the non-negativity restrictions :

$$x_1 \ge 0 , x_2 \ge 0 , x_3 \ge 0. \qquad \text{...(3.2)}$$

The total income from the finished cloth is given by

$$\mathbf{P} = 3x_1 + 5x_2 + 4x_3\,. \qquad \text{...(3.3)}$$

Thus the problem now becomes to find x_1 , x_2 , x_3 satisfying the restrictions (3·1) and (3·2) and maximizing the profit function **P**.

Note. This linear programming problem has been solved by simplex method as *Example* 5 (page 125).

Example 7. *A firm manufactures 3 products A, B and C. The profits are Rs. 3, Rs. 2 and Rs. 4 respectively. The firm has 2 machines and below is the required processing time in minutes for each machine on each product.*

		Product		
		A	B	C
Machine	G	4	3	5
	H	2	2	4

Machine G and H have 2,000 and 2,500 machine-minutes, respectively. The firm must manufacture 100 A's, 200 B's and 50 C's, but no more than 150 A's.

Setup an L.P. problem to maximize profit. Do not solve it.

Formulation. Let x_1 , x_2 , x_3 be the number of products *A* , *B* and *C*, respectively.

Since the profits are Rs. 3, Rs. 2 and Rs. 4 respectively, the total profit gained by the firm after selling these three products is given by $\mathbf{P} = 3x_1 + 2x_2 + 4x_3$.

Now the total number of minutes required in producing these three products at machine *G and H are given by*

$$4x_1 + 3x_2 + 5x_3 , \quad \text{and} \quad 2x_1 + 2x_2 + 4x_3, \text{ respectively.}$$

But, there are only 2,000 minutes available at machine *G* and 2,500 minutes at machine *H* , therefore the restrictions will be

$$4x_1 + 3x_2 + 5x_3 \le 2{,}000 \quad \text{and} \quad 2x_1 + 2x_2 + 4x_3 \le 2{,}500\,.$$

Also, since the firm manufactures 100 *A's* , 200 *B's* and 50 *C's* but not more than 150 *A's*, therefore further restrictions become :

$$100 \le x_1 \le 150, 200 \le x_2 \ge 0 \text{ and } 50 \le x_3 \ge 0\,.$$

Hence the allocation problem of the firm can be finally put in the form :

Find the value of x_1, x_2, x_3 so as to maximize

$$P = 3x_1 + 2x_2 + 4x_3$$

subject to the constraints :

$$4x_1 + 3x_2 + 5x_3 \le 2{,}000$$
$$2x_1 + 2x_2 + 4x_3 \le 2{,}500$$
$$100 \le x_1 \le 150, 200 \le x_2 \ge 0, 50 \le x_3 \ge 0.$$

Example 8. *A farmer has 100 acre farm. He can sell all tomatoes, lettuce, or radishes he can raise. The price he can obtain is Re. 1.00 per kg for tomatoes, Rs. 0·75 a head for lettuce and Rs. 2·00 per kg for radishes. The average yield per-acre is 2,000 kg of tomatoes, 3000 heads of lettuce, and 1000 kgs of radishes. Fertilizer is available at Rs. 0·50 per kg and the amount required per acre is 100 kgs each for tomatoes and lettuce, and 50 kgs for radishes. Labour required for sowing, cultivating and harvesting per acre is 5 man-days for tomatoes and radishes, and 6 man-days for lettuce. A total of 400 man-days of labour are available at Rs. 20·00 per man-day.*

Formulate this problem as a linear programming model to maximize the farmer's total profit.

[Kanpur (B.Sc.) 93, 92]

Formulation. Farmer's problem is to decide how much area should be alloted to each type of crop he wants to grow to maximize his total profit. Let the farmer decide to allot x_1, x_2 and x_3 acre of his land to grow tomatoes, lettuce and radishes respectively. So the farmer will produce $2000x_1$ kgs of tomatoes, $3000x_2$ heads of lettuce, and $1000x_3$ kgs of radishes.

Therefore, total sale will be $=$ Rs. $[2000\,x_1 + 0{\cdot}75 \times 3000\,x_2 + 2 \times 1000\,x_3]$

Fertilizer expenditure will be $=$ Rs. $[0{\cdot}50\,\{100\,(x_1 + x_2) + 50x_3\}]$

Labour expenditure will be $=$ Rs. $[20 \times (5x_1 + 6x_2 + 5x_3)]$

Therefore, farmer's net profit will be

$$P = \text{Total sale (in Rs.)} - \text{Total expenditure (in Rs.)}$$

or $P = [2000\,x_1 + 0{\cdot}75 \times 3000\,x_2 + 2 \times 1000\,x_3] - 0{\cdot}50 \times [100\,(x_1 + x_2) + 50\,x_3] - 20 \times [5x_1 + 6x_2 + 5x_3]$

or $P = 1850\,x_1 + 2080\,x_2 + 1875\,x_3$.

Since total area of the farm is restricted to 100 acre, $x_1 + x_2 + x_3 \le 100$.

Also, the total man-days labour is restricted to 400 man-days, therefore, $5x_1 + 6x_2 + 5x_3 \le 400$.

Hence the farmer's allocation problem can be finally put in the form :

Find the value of x_1, x_2, x_3 so as to maximize :

$$P = 1850\,x_1 + 2080\,x_2 + 1875\,x_3,$$

subject to the conditions :

$$x_1 + x_2 + x_3 \le 100,$$
$$5x_1 + 6x_2 + 5x_3 \le 400,$$

and $x_1, x_2, x_3 \ge 0$.

Example 9. *A manufacturer produces three models (I, II and III) of a certain product. He uses two types of raw material (A and B) of which 4000 and 6000 units repectively are available. The raw material requirements per unit of the three models are given below :*

Raw Material	*Requirement per unit of given model*		
	I	*II*	*III*
A	2	3	5
B	4	2	7

The labour time for each unit of model I is twice that of model II and three times that of model III. The entire labour force of the factory can produce the equivalent of 2500 units of model I. A market survey indicates that the minimum demand of the three models are 500, 500 and 375 units respectively. However, the ratios of the number of units produced must be equal to 3 : 2 : 5. Assume that the profit per unit of models I, II *and III are rupees 60, 40 and 100 respectively. Formulate the problem as a linear programming model in order to determine the number of units of each product which will maximize profit.* **[JNTU (B.Tech) 98]**

Formulation. Let the manufacturer produce x_1, x_2, x_3 units of model I, II and III, respectively. Then, the raw material constraints will be

$$2x_1 + 3x_2 + 5x_3 \leq 4{,}000 \quad \text{(for A)}$$
$$4x_1 + 2x_2 + 7x_3 \leq 6{,}000 \quad \text{(for B)}$$

Suppose it takes labour time t for producing one unit of model I, so by the given condition it will take $t/2$ and $t/3$ labour time for producing one unit of model II and III, respectively.

As the factory can produce 2500 units of model I, so the restriction on the production time will be $tx_1 + (t/2)x_2 + (t/3)x_3 \leq 2500t$, *i.e.*,

$$x_1 + 1/2\, x_2 + 1/3\, x_3 \leq 2500.$$

Also, since at least 500 units of model type I and II each and 375 units of model III are demanded, the constraints of market demand needs,

$$x_1 \geq 500, \quad x_2 \geq 500, \quad \text{and} \quad x_3 \geq 375.$$

But, the ratio of the number of units of different types of models is 3 : 2 : 5, we have $1/3\, x_1 = 1/2\, x_2$ and $1/2\, x_2 = 1/5\, x_3$.

Since the profit per unit on model I, II and III are Rs. 60, Rs. 40 and Rs. 100 respectively, the objective function is to maximize the profit : $P = 60x_1 + 40x_2 + 100x_3$.

Thus, the linear programming problem is :
To maximize : $P = 60x_1 + 40x_2 + 100x_3$,
subject to the constraints :

$$2x_1 + 3x_2 + 5x_3 \leq 4{,}000$$
$$4x_1 + 2x_2 + 7x_3 \leq 6{,}000$$
$$x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 \leq 2{,}500$$
$$\frac{1}{3}x_1 = \frac{1}{2}x_2, \frac{1}{2}x_2 = \frac{1}{5}x_3$$

and $x_1 \geq 500, x_2 \geq 500, x_3 \geq 375.$

Example 10. (Diet Problem.) *One of the interesting problems in linear programming is that of balanced diet. Dieticians tell us that a balanced diet must contain quantities of nutrients such as calories, minerals, vitamins, etc. Suppose that we are asked to find out the food that should be recommended from a large number of alternative sources of these nutrients so that the total cost of food satisfying minimum requirements of balanced diet is the lowest.*

The medical experts and dieticians tell us that it is necessary for an adult to consume at least 75 g of proteins, 85 g of fats, and 300 g of carbohydrates daily. The following table gives the food items (which are readily available in the market), analysis, and their respective costs.

Food type	Food value (gms.) per 100g			Cost per kg. (Rs.)
	Proteins	Fats	Carbohydrates	
1	8.0	1.5	35.0	1.00
2	18.0	15.0	—	3.00
3	16.0	4.0	7.0	4.00
4	4.0	20.0	2.5	2.00
5	5.0	8.0	40.0	1.50
6	2.5	—	25.0	3.00
Minimum daily requirements	75	85	300	

Formulation. Let x_1, x_2, x_3, x_4, x_5 and x_6 units of food types respectively be used per day in a diet and the total diet must at least supply the minimum requirements. The object is to minimize total cost C of diet. The objective function thus becomes

$$z = x_1 + 3x_2 + 4x_3 + 2x_4 + 1{\cdot}5\, x_5 + 3x_6.$$

Since 8, 18, 16, 4, 5 and 2.5 gms of proteins are available from 100 gm unit of each type of food respectively, total proteins available from x_1, x_2, x_3, x_4, x_5 and x_6 units of each food respectively will be $8x_1 + 18x_2 + 16x_3 + 4x_4 + 5x_5 + 2.5x_6$ gms. daily.

But minimum daily requirement of proteins as prescribed is 75 gms. Hence, the *protein* requirement constraint is

$$8x_1 + 18x_2 + 16x_3 + 4x_4 + 5x_5 + 2{\cdot}5x_6 \geq 75 \; (Protein)$$

Similarly, *fats* and *carbohydrates* requirement constraints are obtained respectively as given below :

$$1.5x_1 + 15x_2 + 4x_3 + 20x_4 + 8x_5 + 0x_6 \geq 85 \; (Fats)$$

$$35x_1 + 0x_2 + 7x_3 + 2.5x_4 + 40x_5 + 25x_6 \geq 300 \; (Carbohydrates)$$

Further, $x_1, x_2, x_3, x_4, x_5, x_6$ are all non-negative quantities, *i.e.*

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0.$$

Generalization. A generalization of this problem is as follows :

Let a_{ij} be the number of units of nutrient i in one unit of food j, x_j be the units of food j used per day, and b_i be the requirement of the ith nutrient. Thus, the objective function becomes :

Minimize $z = c_1x_1 + c_2x_2 + \ldots c_nx_n$ subject to the nutrient requirements :

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geq b_m$$
$$x_1, x_2, x_3, \ldots, x_n \geq 0,$$

where n defines the number of food items, m the different nutrients, and c_j the cost per unit of food j $(j = 1, 2, \ldots, n)$.

It should be noted that diet problem as formulated above does not provide variation in the diet. The same combination must be taken every day (to keep ones health good). It is assumed that the diet suggested is palatable (let us hope so). The constraints place the bound on the minimum size of the food value and it may turnout that a person may be taking too much of carbohydrate while trying to keep the protein requirements (too bad for fattys). Hence an upper bound would also be desirable if the intake beyond limit is bad for health.

Example 11. *A manufacturer of biscuits is considering four types of gift packs containing three types of biscuits : orange cream (OC), chocolate cream (CC), and wafers (W). Market research conducted recently to assess the preferences of the consumers shows the following types of assortments to be in good demand :*

Assortments	Contents	Selling price per kg. (Rs.)
A	Not less than 40% of OC, not more than 20% of CC, any quantity of W	20
B	Not less than 20% of OC, not more than 40% of CC, any quantity of W	25
C	Not less than 50% of OC, not more than 10% of CC, any quantity of W	22
D	No restrictions	12

For the biscuits, the manufacturing capacity and costs are given below :

Biscuits variety	:	*OC*	*CC*	W
Plant capacity (kg/day)	:	200	200	150
Manufacturing cost (Rs./kg)	:	8	9	7

Formulate a linear programming model to find the production schedule which maximizes the profit assuming that there are no market restrictions.

Formulation. Let the decision variables x_{ij} $(i = A, B, C, D; j = 1, 2, 3)$ be defined as follows :

(i) For the gift pack A, x_{A1}, x_{A2}, x_{A3} denote the quantity in kg. of OC, CC, and W type of biscuits;

(ii) For the gift pack B, x_{B1} x_{B2}, x_{B3} denote the quantity in kg. of OC, CC, and W type of biscuits;

(iii) For the gift pack C, x_{C1}, x_{C2}, x_{C3} denote the quantity in kg. of OC, CC, and W type of biscuits;

(iv) For the gift pack D, x_{D1}, x_{D2}, x_{D3} denote the quantity in kg. of OC, CC, and W type of biscuits;

Now the given data can be put in the form of linear programming problem as follows :
Maximize, $\mathbf{P} = 20\,(x_{A1} + x_{A2} + x_{A3}) + 25\,(x_{B1} + x_{B2} + x_{B3}) + 22\,(x_{C1} + x_{C2} + x_{C3}) + 12\,(x_{D1} + x_{D2} + x_{D3})$

$$- 8\,(x_{A1} + x_{B1} + x_{C1} + x_{D1}) - 9\,(x_{A2} + x_{B2} + x_{C2} + x_{D2}) - 7\,(x_{A3} + x_{B3} + x_{C3} + x_{D3})$$

$$= 12\,x_{A1} + 11x_{A2} + 13x_{A3} + 17x_{B1} + 16x_{B2} + 18x_{B3} + 14x_{C1} + 13x_{C2} + 15x_{C3} + 4x_{D1} + 3x_{D2} + 5x_{D3};$$

subject to the constraints :

Gift pack A : $x_{A1} \geq 0.40\,(x_{A1} + x_{A2} + x_{A3}), x_{A2} \leq 0.20\,(x_{A1} + x_{A2} + x_{A3})$

Gift pack B : $x_{B1} \geq 0.20\,(x_{B1} + x_{B2} + x_{B3}), x_{B2} \leq 0.40\,(x_{B1} + x_{B2} + x_{B3})$

Gift pack C : $x_{C1} \geq 0.50\,(x_{C1} + x_{C2} + x_{C3}), x_{C2} \leq 0.10\,(x_{C1} + x_{C2} + x_{C3})$

Plant capacity constraints are :

$$x_{A1} + x_{B1} + x_{C1} + x_{D1} \leq 200, x_{A2} + x_{B2} + x_{C2} + x_{D2} \leq 200, x_{A3} + x_{B3} + x_{C3} + x_{D3} \leq 150$$

$$x_{ij} \geq 0 \text{ (for } i = A, B, C, D \text{ and } j = 1, 2, 3).$$

Example 12. *A complete unit of a certain product consists of four units of component A and three units of component B. Two components (A and B) are manufactured from two different raw materials of which 100 units and 200 units, respectively, are available. Three departments are engaged in the production process with each department using a different method for manufacturing the components. The following table gives the raw material requirements per production run and the resulting units of each component. The objective is to determine the number of production runs for each department which will maximize the total number of component units of the final product.*

Department	*Input per run (units)*		*Out put per run (units)*	
	Raw material I	*Raw material II*	*Component A*	*Component B*
1	7	5	6	4
2	4	8	5	8
3	2	7	7	3

Formulation. Let x_1, x_2, x_3 be the number of production runs for the departments 1,2,3 respectively.

The total number of units produced by three departments :

$$6x_1 + 5x_2 + 7x_3 \text{ (component } A), \qquad 4x_1 + 8x_2 + 3x_3 \text{ (component } B)$$

The restrictions on the raw materials I and II are, respectively, given by

$$7x_1 + 4x_2 + 2x_3 \leq 100 \quad \text{and} \quad 5x_1 + 8x_2 + 7x_3 \leq 200.$$

Since the objectiv function is to maximize the total number of units of the final product and each such unit requires 4 units of component A and 3 units of component B, the maximum number of units of the final product cannot exceed the smaller value of

$$\tfrac{1}{4}\,(6x_1 + 5x_2 + 7x_3) \quad \text{and} \quad \tfrac{1}{3}\,(4x_1 + 8x_2 + 3x_3).$$

The objective function thus becomes :

Maximize $\quad z = \min\,[\tfrac{1}{4}\,(6x_1 + 5x_2 + 7x_3), \tfrac{1}{3}\,(4x_1 + 8x_2 + 3x_3)]$

Since this objective function is not linear, a suitable transformation can be used to reduce the above model to an acceptable linear programming format.

Suppose, $\quad \min.\,[\tfrac{1}{4}\,(6x_1 + 5x_2 + 7x_3), \tfrac{1}{3}\,(4x_1 + 8x_2 + 3x_3)] = v.$

Therefore, $\quad \tfrac{1}{4}\,(6x_1 + 5x_2 + 7x_3) \geq v \quad \text{and} \quad \tfrac{1}{3}\,(4x_1 + 8x_2 + 3x_3) \geq v.$

In fact, at least one of these two inequalities must hold as an equation in any solution because the number of final assembly units, v, is maximized. Then, its upper limit is specified by the smaller of the left hand sides of above two inequalities. This indicates that the two inequalities are equivalent to the original equation defining v.

Now the above problem can be put into the following linear programming form : maximize $z = v$, subject to the constraints :

$$6x_1 + 5x_2 + 7x_3 - 4v \geq 0, \quad 4x_1 + 8x_2 + 3x_3 - 3v \geq 0,$$

$$7x_1 + 4x_2 + 2x_3 \leq 100, \quad 5x_1 + 8x_2 + 7x_3 \leq 200, \quad \text{and} \quad x_1, x_2, x_3, v \geq 0.$$

Example 13. *A leading C.A. is attempting to determine a 'best' investment portfolio and is considering six alternative investment proposals. The following table indicates point estimates for the price per share, the annual growth rate in the price per share, the annual dividend per share and a measure of the risk associated with each investment.*

Portfolio Data

Shares under consideration :	A	B	C	D	E	F
Current price per share (Rs.)	80	100	160	120	150	200
Projected annual growth rate	0·08	0·07	0·10	0·12	0·09	0·15
Projected annual dividend per share (Rs.)	4·00	4·50	7·50	5·50	5·75	0·00
Projected risk in return	0·05	0·03	0·10	0·20	0·06	0·08

The total amount available for investment is Rs. 25 lakhs and the following conditions are required to be satisfied.

(i) The maximum rupee amount to be invested in alternative F is Rs. 2,50,000.

(ii) No more than Rs. 5,00,000 should be invested in alternatives A and B combined.

(iii) Total weighted risk should not be greater than 0·10, where

$$\text{Total weighted risk} = \frac{\text{(Amount invested in alternative j) (Risk of alternative j)}}{\text{Total amount invested in all the alternatives}}$$

(iv) For the sake of diversity, at least 100 shares of each stock should be purchased.

(v) At least 10 per cent of the total investment should be in alternatives A and B combined.

(vi) Dividends for the year should be at least 10,000.

Rupees return per share of stock is defined as price per share one year hence less current price per share PLUS dividend per share. If the objective is to maximize total rupee return, formulate the linear programming model for determining the optimal number of shares to be purchased in each of the shares under consideration. You may assume that the time horizon for the investment is one year. The formulated LP problem is not required to be solved. **[C.A. (Final) Nov. 91]**

Formulation. Let x_1, x_2, x_3, x_4, x_5 and x_6 represent the number of shares to be purchased in each of the six investment proposals A, B, C, D, E and F.

Rupee return per share= Price per share one year hence – current price per share + dividend per share)
= Current price per share × Projected annual growth rate
(*i.e.* Project growth each year)+ dividend per share

Thus, we compute the following data :

Investment Alternatives	:	A	B	C	D	E	F
No. of shares purchased	:	x_1	x_2	x_3	x_4	x_5	x_6
Projected growth for each share (Rs.)	:	6·40	7·00	16·00	14·40	13·50	30·00
Projected annual dividend per share (Rs.)	:	4·00	4·50	7·50	5·50	5·75	0·00
Rupee return per share	:	10·40	11·50	23·50	19·90	19·25	30·00

The Chartered Accountant wants to maximize the total rupee return, thus the objective function of the linear programming problem is given by :

$$\text{Maximize } R = 10{\cdot}40x_1 + 11{\cdot}50x_2 + 23{\cdot}50x_3 + 19{\cdot}90x_4 + 19{\cdot}25x_5 + 30{\cdot}00x_6,$$

subject to the constraints (1) to (7) as stated below.

Since the total amount available for investment is Rs. 25 lakhs, therefore

(1) $80x_1 + 100x_2 + 160x_3 + 120x_4 + 150x_5 + 200x_6 \leq 25{,}00{,}000$

(2) $200x_6 \leq 2{,}50{,}000$ [from condition (1)]

(3) $80x_1 + 100x_2 \leq 5{,}00{,}000$ [from condition (2)]

(4) According to condition (3) of the problem

$$\left[\frac{80x_1\,(0{\cdot}05) + 100x_2\,(0{\cdot}03) + 160x_3\,(0{\cdot}10) + 120x_4(0{\cdot}20) + 150x_5\,(0{\cdot}06) + 200x_6(0{\cdot}08)}{80x_1 + 100x_2 + 160x_3 + 120x_4 + 150x_5 + 200x_6}\right] \leq 0{\cdot}10$$

or $\qquad 4x_1 + 3x_2 + 16x_3 + 24x_4 + 9x_5 + 16x_6 \leq 8x_1 + 10x_2 + 16x_3 + 12x_4 + 15x_5 + 20x_6$

or $\qquad -4x_1 - 7x_2 + 0x_3 + 12x_4 - 6x_5 - 4x_6 \leq 0$

(5) $x_1 \geq 100, x_2 \geq 100, x_3 \geq 100, x_4 \geq 100, x_5 \geq 100, x_6 \geq 100$ [from condition (4)]

(6) $80x_1 + 100x_2 \geq 0{\cdot}10\,(80x_1 + 100x_2 + 160x_3 + 120x_4 + 150x_5 + 200x_6)$ [from condition (5)]

or $\qquad 80x_1 + 100x_2 \geq 8x_1 + 10x_2 + 16x_3 + 12x_4 + 15x_5 + 20x_6$

or $\qquad 72x_1 + 90x_2 \quad 16x_3 - 12x_4 - 15x_5 - 20x_6 \geq 0$

(7) $4x_1 + 4{\cdot}5x_2 + 7{\cdot}5x_3 + 5{\cdot}5x_4 + 5{\cdot}75x_5 \geq 10{,}000$ [from condition (6)]

Finally, combining all the constraints from (1) to (7), the desired linear programming problem is formulated.

Example 14. *Consider a company that must produce two products over a production period of three months of duration. The company can pay for materials and labour from two sources : company funds and borrowed funds.*

The firm faces three decisions :

(1) How many units should it produce of Product 1 ?

(2) How many units should it produce of Product 2 ?

(3) How much money should it borrow to support the production of the two products ?

In making these decisions, the firm wishes to maximize the profit contribution subject to the conditions stated below :

(i) Since the company's products are enjoying a seller's market, it can sell as many units as it can produce. The company would therefore like to produce as many units as possible subject to production capacity and financial constraints. The capacity constraints, together with cost and price data, are given in Table–1.

Table–1 : Capacity, Price and Cost data

Product	Selling Price (Rs. per unit)	Cost of Productionj (Rs. per unit)	Required Hours per unit in Department		
			A	B	C
1	14	10	0·5	0·3	0·2
2	11	8	0·3	0·4	0·1
Available hours per production period of three months :			500	400	200

(ii) The available company funds during the production period will be Rs. 3 lakhs.

(iii) A bank will give loans upto Rs. 2 lakhs per production period at an interest rate of 20 per cent per annum provided the company's acid (quick) test ratio is at least 1 to 1 while the loan is outstanding. Take a simplified acid-test ratio given by

$$\frac{\textit{Surplus cash on hand after production} + \textit{Accounts receivable}}{\textit{Bank borrowings} + \textit{Interest occurred there on}}$$

(iv) Also make sure that the needed funds are made available for meeting the production costs.

Formulate the above as a linear programming problem. **[C.A. (Nov. 92)]**

Formulation. Let x_1 = no. of units of product 1 produced, x_2 = no. of units of product 2 produced, and x_3 = amount of money borrowed.

The profit contribution per unit of each product is given by (selling price – variable cost of production). Total profit can be computed by [summing of the profits from producing the two products – the cost associated with borrowed funds (if any)]. The objective function is thus obtained as :

Maximize $\qquad \mathbf{P} = (14 - 10)\,x_1 + (11 - 8)\,x_2 - 0{\cdot}05\,x_3 = 4x_1 + 3x_2 - 0{\cdot}05x_3$

(since the interest rate is 20% per annum, hence it will be 5% for a period of three months.)

Subject to the following constraints :

The production capacity constraints for each department (as given by table 1) are :

$0{\cdot}5x_1 + 0{\cdot}3x_2 \leq 500$ ...(1) $\qquad 0{\cdot}3x_1 + 0{\cdot}4x_2 \leq 400$ ...(2) $\qquad 0{\cdot}2x_1 + 0{\cdot}1x_2 \leq 200$ ...(3)

The funds for available production include both Rs. 3,00,000 cash that the firm possesses and any borrowed funds maximum up to Rs. 2,00,000. Consequently, production is limited to the extent that funds are available to pay for production costs. The constraint representing this relationship is given by :

Funds required for production ≤ Funds available

i.e. $10x_1 + 8x_2 \leq \text{Rs. } 3{,}00{,}000 + x_3$ or $10x_1 + 8x_2 - x_3 \leq \text{Rs. } 3{,}00{,}000$ ...(4)

The borrowed funds constraint [from condition (iii) of the problem] is given by $x_3 \leq$ Rs. 2,00,000

The constraint based on the acid-test condition is developed as follows :

$$\frac{\text{Surplus cash on hand after production + Accounts receivable}}{\text{Bank borrowings + Interest occured there on}} \geq 1$$

i.e.
$$\frac{(3{,}00{,}000 + x_3 - 10x_1 - 8x_2) + 14x_1 + 11x_2}{x_3 + 0{\cdot}05x_3} \geq 1$$

or
$$3{,}00{,}000 + x_3 + 4x_1 + 3x_2 \geq (x_3 + 0{\cdot}05x_3)$$

or
$$-4x_1 - 3x_2 + 0{\cdot}05x_3 \leq 3{,}00{,}000 \quad ...(5)$$

Finally the linear programming problem is formulated as :

Max. $\mathbf{P} = 4x_1 + 3x_2 - 0{\cdot}05x_3$, subject to the constraints :

$$0{\cdot}5x_1 + 0{\cdot}3x_2 \leq 500, 0{\cdot}3x_1 + 0{\cdot}4x_2 \leq 400, 0{\cdot}2x_1 + 0{\cdot}1x_2 \leq 200$$

$$10x_1 + 8x_2 - x_3 \leq \text{Rs. } 3{,}00{,}000, x_3 \leq \text{Rs. } 2{,}00{,}000$$

$$-4x_1 - 3x_2 + 0{\cdot}05x_3 \leq \text{Rs. } 3{,}00{,}000, \text{ where } x_1, x_2, x_3 \geq 0.$$

Example 15. *WELLTYPE Manufacturing company produces three types of typewriters. All the three models are required to be machined first then assembled. The time required for various models are as follows :*

Types	Manual typewriters	Electronic typewriters	Deluxe elec. typewriters
Machine time (in hours)	15	12	14
Assembly time (in hours)	4	3	5

The total available machine time and assembly time are 3000 hours and 1,200 hours, respectively. The data regarding the selling price and variable costs for the three types are :

Type	Manual	Electronic	Deluxe Electronic
Selling price (Rs.)	4,100	7,500	14,600
Labour, material and other variable costs (Rs.)	2,500	4,500	9,000

The company sells all the three types on credit basis, but will collect the amounts on the first of next month. The labour, material and other variable expenses will have to be paid in cash. This company has taken a loan of Rs. 40,000 from a co-operative bank and this company will have to repay it to the bank on Ist April, 1993. The TNC Bank from whom this company has borrowed Rs. 60,000 has expressed its approval to renew the loan.

The Balance Sheet of this company as on 31.3.93 is as follows :

Liabilities	Rs.	Assets	Rs.
Equity share capital	1,50,000	Land	90,000
Capital reserve	15,000	Building	70,000
General Reserve	1,10,000	Plant & Machinery	1,00,000
Profit & Loss a/c	25,000	Furniture & Fixures	15,000
Long term loan	1,00,000	Vehicles	30,000
Loan from TNC Bank	60,000	Inventory	5,000
Loan from Co-op. Bank	40,000	Receivables	50,000
		Cash	1,40,000
Total	5,00,000	Total	5,00,000

The company will have to pay a sum of Rs. 10,000 towards the salary from top management executives and other fixed overheads for the month. Interest on long term loans is to be paid every month at 24% per annum. Interest on loans from TNC and Co-operative Banks may be taken to be Rs. 1,200 for the month. Also this company has promised to deliver 2 Manual typewriters and 8 Deluxe-Electronic type writers to one of its valued customers next month. Also make sure that the level of operations in this company is subject to the availability of cash next month. This company will also be able to sell all their types of type writers in the market. The senior manager of this company desires to know as to how many units of each typewriter must be manufactured in the factory next month so as to maximize the profits of the company.

Formulate this as a linear programming problem and need not to be solved. **[C.A. (May) 93]**

Formulation. Let x_1, x_2 and x_3 be the number of *Manual, Electronic* and *Deluxe-Electronic* typewriters respectively which are to be manufactured in the factory next month. From the given data, profit contribution per unit in rupees will be (4,100–2500), (7,500–4,500) and (14,600–9,000) for Manual, Electronic and Deluxe–Elec. typewriters, respectively, *i.e.* Rs. 1600, Rs. 3000 and Rs. 5,600 respectively. Therefore, the objective function is given by :

$$\text{Maximize } P = 1600x_1 + 3{,}000x_2 + 5{,}600x_3.$$

From the data given for the time required for various models, we get the following constraints :

$$15x_1 + 12x_2 + 14x_3 \le 3{,}000 \text{ (machine time restriction)}$$

and

$$4x_1 + 3x_2 + 5x_3 \le 1{,}200 \text{ (assembly time restriction)}$$

The level of operations in the company is subject to the availability of cash next month.

The cash requirements for x_1 units of Manual, x_2 units of Electronic and x_3 units of Deluxe-Electronic typewriters are :

$$2{,}500x_1 + 4{,}500x_2 + 9{,}000x_3 \qquad ...(1)$$

The cash availability for the next month from the balance sheet is as follows :

Cash availability (Rs.) = Cash balance (Rs. 1,40,000) + Receivables (Rs. 50,000)
– Loan to repay to co-operative bank (Rs. 40,000)
– Interest on loan from TNC & Co-op banks (Rs. 1200)
– Interest on long term loans $\left(\dfrac{0{\cdot}24 \times \text{Rs. } 1{,}00{,}000}{12}\right)$
– Top management salary & fixed overheads Rs. 10,000

That is, Cash availability = Rs. 1,40,000 + Rs. 50,000 – Rs. (40,000 + 1200 + 2000 + 10,000)
= Rs. 1,90,000 – Rs. 53,200 = Rs. 1,36,800 ...(2)

From (1) and (2), we get the constraint

$$2{,}500\,x_1 + 4{,}500\,x_2 + 9{,}000\,x_3 \le 1{,}36{,}800.$$

Also, the company has promised to deliver 2 Manual and 8 Deluxe-Elec. typewriters to one on its customers. Hence $x_1 \ge 2, x_2 \ge 0$, and $x_3 \ge 8$.

Finally, the formulated LPP can be put in the following form.

$$\text{Max. } P = 1{,}600x_1 + 3{,}000x_2 + 5{,}600x_3, \text{ subject to the constraints :}$$

$$15x_1 + 12x_2 + 14x_3 \le 3000,\ 4x_1 + 3x_2 + 5x_3 \le 1{,}200,\ 2500x_1 + 4500x_2 + 9000x_3 \le 1{,}36{,}800$$

$$x_1 \ge 2, x_2 \ge 0, x_3 \ge 8 \text{ and } x_1, x_2, x_3 \text{ can take positive integral values only.}$$

Example 16. *The most recent audited summarized Balance Sheet of Shop and Shop Financial services is given below :*

Balance Sheet as on March 31, 1994

Liabilities	(Rs. in lakhs)	Assets	(Rs. in lakhs)
Equity Share Capital	65	**Fixed Assets :**	
Reserves & Surplus	110	Assets on Lease (original cost : Rs. 550 lakhs)	375
Term Loan from IFCI	80	Other Fixed Assets	50
Public Deposites	150	Investments (on wholly owned subsidiaries)	20
Bank Borrowings	147	**Current Assets :**	
Other Current Liabilities	50	Stock on Hire	80
		Receivables	30
		Other Current Assets	35
		Miscellaneous expenditure (not written off)	12
Total	602	Total	602

The company intends to enhance its investment in the lease portfolio by another Rs. 1,000 lakhs. For this purpose it would like to raise a mix of debt and equity in such a way that the overall cost of raising additional funds is minimized. The following constraints apply to the way the funds can be mobilized :

(1) Total debt divided by net owned funds, cannot exceed 10.

(2) Amount borrowed from financial institutions cannot exceed 25% of the net worth.

(3) Maximum amount of bank borrowings cannot exceed three times the net owned funds.

(4) The company would like to keep the total public deposit limited to 40% of the total debt.

The post-tax costs of the different sources of finance are as follows :

Equity	Term Loans	Public Deposits	Bank Borrowings
2·5%	8·5%	7%	10%

Formulate the funding problem as LPP.

Note. (a) Total Debt = Term Loans from Financial Institutions + Public Deposites + Bank Borrowings.

(b) Net Worth = Equity Share Capital + Reserves & Surplus.

(c) Net Owned Funds = Net Worth − Miscellaneous Expenditures. **[CA. (May, 94)]**

Formulation. Let x_1, x_2, x_3 and x_4 be the quantity of additional funds (in lakhs) raised on account of additional equity, additional term loans, additional public deposites and additional bank borrowings, respectively. The objective function is to minimize the cost of additional funds raised by the company. That is,

Minimize $C = 0{\cdot}25x_1 + 0{\cdot}85x_2 + 0{\cdot}07x_3 + 0{\cdot}1x_4$, subject to the following constraints :

(1) $\left[\dfrac{\text{Total Debt}}{\text{Net owned funds}} \le 10\right]$ or $\left[\dfrac{\text{(Existing debt + Additional)}}{\text{(Equity share capital + Reserve \& Surplus + Additional Equity − Misc. Exp.)}}\right] \le 10$

or $\dfrac{80 + 150 + 147 + x_2 + x_3 + x_4}{(65 + 110 + x_1) - 12} \le 10$ or $\dfrac{x_2 + x_3 + x_4 + 377}{x_1 + 163} \le 10$

or $x_2 + x_3 + x_4 + 377 \le 10x_1 + 1630$ or $-10x_1 + x_2 + x_3 + x_4 \le 1253.$

(2) Amount borrowed (financial institutions) ≤ 25% of net worth

or (Existing long term loan from financial institutions + Additional loan) ≤ 25% (Existing Equity Capital + Reserve & Surplus + Addl. Equity Capital)

or $80 + x_2 \le 0{\cdot}25\,(175 + x_1)$ or $80 + x_2 \le \frac{1}{4}(175 + x_1)$

or $320 + 4x_2 \le 175 + x_1$ or $-x_1 + 4x_2 \le -145$ or $x_1 - 4x_2 \ge 145$

(3) Maximum bank borrowings ≤ 3 (Net owned funds)

or (Existing bank borrowings + Addl. bank borrowings) ≤ 3 {(Existing Equity Capital + Reserves & Surplus + Addl. Equity Capital − Misc. Exp.}

or $(147 + x_4) \le 3\,(65 + 110 + x_1 - 12)$ or $x_4 - 3x_1 \le 525 - 36 - 147$ or $-3x_1 + x_4 \le 342.$

(4) Total public deposit ≤ 40% of total debt

or (Existing public deposite + addl. public deposits) ≤ 0·40 (Existing total debt + Addl. total debt.)

or $150 + x_3 \le 0{\cdot}40\,(80 + 150 + 147 + x_2 + x_3 + x_4)$ or $150 + x_3 \le 0{\cdot}40\,(x_2 + x_3 + x_4 + 377)$

or $1500 + 10x_3 \le 4x_2 + 4x_3 + 4x_4 + 1508$ or $-4x_2 + 6x_3 - 4x_4 \le 8.$

(5) Addl. equity capital + Addl. term loan + Addl. public deposits + Addl. bank borrowings = 1000 (since the company wants to enhance the investment by Rs. 1,000 lakhs)

or $x_1 + x_2 + x_3 + x_4 = 1000$

(6) $x_1, x_2, x_3, x_4 \ge 0.$

Example 17. *Renco- Foundries is in the process of drawing up a Capital Budget for the next three years. It has funds to the time of Rs. 1,00,000 which can be allocated across the projects A, B, C, D and E. The net cash flows associated with an investment of Re. 1 in each project are provided in the following table.*

	Cash Flow at Time			
	0	1	2	3
From inv. A	− Re. 1	+ Re. 0·5	+ Rs. 1	Re. 0
From inv. B	Re. 0	− Re. 1	+ Re. 0·5	+ Re 1
From inv. C	− Re. 1	+ Rs. 1·2	Re. 0	Re. 0
From inv. D	− Re. 1	Re. 0	Re. 0	Rs. 1·9
From inv. E	Re. 0	Re. 0	− Re. 1	Rs. 1·5

Note : Time 0 = present, Time 1 = 1 year from now, Time 2 = 2 years from now, Time 3 = 3 years from now.

For example, Re. 1 invested in investment B requires a Re. 1 cash outflow at time 1 and returns Re. 0·50 at time 2 and Re. 1 at time 3.

To ensure that the firm remains reasonably diversified, the firm will not commit an investment exceeding Rs. 75,000 in any project. The firm cannot borrow funds; therefore the cash available for investment at any *time is limited to cash on hand. The firm will earn interest at 8% per annum by parking the uninvested funds in money market investments. Assume that the returns from investments can be immediately re-invested. For example, the positive cash flow received from project C at time 1 can immediately be reinvested in project B.*

Required : *Formulate an L.P. that will "Maximize cash on hand at time 3".* **[C.A. (Nov. 95)]**

Formulation. The company wants to decide optimum allocation of funds to project A, B, C, D, E and money market investments.

Let x_1, x_2, x_3, x_4 and x_5 be the amount of rupees invested in investments A, B, C, D and E respectively and s_i be the amount of Rs. invested in Money market investments at time i (for $i = 0, 1, 2$).

The objective of the company is to draw up the capital budget in such a way that will "maximize cash on hand at time 3", At time 3, the cash on hand for company will be the sum of all cash inflows at time 3.

Since the firm earns interest at 8% per annum by parking the uninvested funds in money market investments, hence Rs. s_0, Rs. s_1 and Rs. s_2 which are invested in these investments at times 0, 1 and 2 will become $1{\cdot}08\, s_0$, $1{\cdot}08 s_1$ and $1{\cdot}08 s_2$ at times 1, 2 and 3, respectively.

From the given table, it can be computed that at time 3 :

Cash on hand $= x_1 \times \text{Rs. } 0 + x_2 \times \text{Re } 1 + x_3 \times \text{Re } 0 + x_4 \times \text{Rs. } 1{\cdot}9 + x_5 \times \text{Rs. } 1{\cdot}5 + 1{\cdot}08 s_2$

$= \text{Rs. } (x_2 + 1{\cdot}9x_4 + 1{\cdot}5x_5 + 1{\cdot}08\, s_2)$

The objective of the company is to maximize the cash at time 3. Hence the objective function will be :

$$\text{Max. } z = x_2 + 1{\cdot}9x_4 + 1{\cdot}5x_5 + 1{\cdot}08 s_2 \quad \text{...(1)}$$

It may be remembered that :

$$\text{Cash available for investment at time } t = \text{ cash on hand at time } t \quad \text{...(2)}$$

At time 0, funds to the tune of Rs. 1,00,000 are available for investment. From the table, it can be observed that funds are invested in investments A, C and D at time 0. Therefore,

$$x_1 + x_3 + x_4 + s_0 = 1{,}00{,}000 \quad \text{...(3)}$$

At time 1, Rs. $0{\cdot}5x_1$, Rs. $1{\cdot}2x_3$ and Rs. $1{\cdot}08\, s_0$ will be available as a result of investments made at time 0. From the table Rs. x_2 and Rs. s_1 are invested in investment B and money market investments, respectively at time 1.

Using (2), we get $$0{\cdot}5x_1 + 1{\cdot}2x_3 + 1{\cdot}08\, s_0 = x_2 + s_1 \quad \text{...(4)}$$

At time 2, Re $1\,x_1$, Rs. $0{\cdot}5x_2$ and Rs. $1{\cdot}08\, s_1$ will be available for investment. However, Rs. x_5 and Rs. s_2 are invested at time 2.

Hence, $$1\,x_1 + 0{\cdot}5x_2 + 1{\cdot}08\, s_1 = x_5 + s_2 \quad \text{...(5)}$$

Also, since the company will not commit an investment exceeding Rs. 75,000 in any project, we get the following constraints :

$$x_i \leq 75{,}000 \text{ for } i = 1, 2, 3, 4, 5. \quad \text{...(6)}$$

and x_1, x_2, x_3, x_4, x_5, s_0, s_1, s_2 are all ≥ 0. Finally, combining all above constraints, the L.P. model for the Renco Foundries is obtained as given below.

$$\text{Max. } z = x_2 + 1{\cdot}9x_4 + 1{\cdot}5x_5 + 1{\cdot}08s_2, \text{ subject to the constraints :}$$
$$x_1 + x_3 + x_4 + s_0 = 1{,}00{,}000,$$
$$0{\cdot}5x_1 + 1{\cdot}2x_3 + 1{\cdot}08s_0 = x_2 + s_1$$
$$1x_1 + 0{\cdot}5x_2 + 1{\cdot}08s_1 = x_5 + s_2$$
$$x_i \leq 75{,}000 \ (i = 1, 2, 3, 4, 5)$$
$$\text{and} \quad x_1, x_2, x_3, x_4, x_5, s_0, s_1, s_2 \text{ are all } \geq 0.$$

Example 18. *An agriculturist has a farm with 126 acres. He produces Radish, Muttar and Potato. Whatever he raises is fully sold in the market. He gets Rs. 5 for Radish per kg., Rs. 4 for Muttar per kg. and Rs. 5 for Potato per kg. The average yield is 1,500 kg. of Radish per acre, 1,800 kg. of Matter per acre and 1,200 kg. of Potato per acre. To produce each 100 kg. of Radish and Muttar and to produce each 80 kg. of Potato, a sum of Rs. 12·50 has to be used for manure. Labour required for each acre to raise the crop is 6 man-days for Radish and Potato each and 5 man-days for Muttar. A total of 500 man-days of labour at a rate of Rs. 40 per man-day are available. Formulate this as a linear programming model to maximize the agriculturist's total profit.*

[C.A. May 97]

Solution. Let x_1, x_2, x_3 be the number of acres allotted for cultivating *Radish, Matter* and *Potato* respectively. Then the following table can be worked out.

	Radish	Muttar	Potato
Selling Price	5 × 1,500 = 7,500	4 × 1,800 = 7,200	5 × 1,200 = 6,000
Manure cost	12·50 × 15 = 187·50	12·50 × 18 = 225	1250 × 15 = 187·50
Labour cost	40 × 6 = 240	40 × 5 = 200	40 × 6 = 240
Profit	7,500 − 427·50 = 7,072·50	7,200 − 425 = 6,775	6,000 − 427·50 = 5,572·50

The linear programming formulation of the given problem is as follows :

Max. (Total profit) $P = 7072{\cdot}5\,x_1 + 6775\,x_2 + 5572{\cdot}5\,x_3$,

subject to the constraints

$$x_1 + x_2 + x_3 \le 125 \quad \textit{(land constraint)}$$

$$6x_1 + 5x_2 + 6x_3 \le 500 \quad \textit{(labour constraint)}$$

$$x_1, x_2, x_3 \ge 0.$$

Example 19. *A company has three operational departments (weaving, processing and packing) with capacity to produce three different types of cloths namely suitings, shirtings, and woolens yielding the profit of Rs. 2, Rs. 4 and Rs. 3 per meter respectively. One meter suiting requires 3 minutes in weaving, 2 minutes in processing and 1 minute in packing 1 meter of shirting requires 4 minutes in weaving, 1 minute in processing and 3 minutes in packing while one meter woolen requires 3 minutes in each department. In a week, total run time of each department is 60, 40 and 80 hours of weaving, processing and packing departments respectively.*

Formulate the linear programming problem to find the product mix to maximize the profit. **[C.A. Nov. 98]**

Solution. The problem may be expressed as follows :

Reserouces / Constraints	Product			Total availability (minutes)
	Suiting	Shirting	Woolen	
Weaving deptt.	3	4	3	60 × 60
Processing deptt.	2	1	3	40 × 60
Packing deptt.	1	3	3	80 × 60

The linear programming formulation can be easily obtained as follows :

Max. (Total profit) $P = 2x_1 + 4x_2 + 3x_3$,

subject to the constraints :

$$3x_1 + 4x_2 + 3x_3 \le 3{,}600$$

$$2x_1 + x_2 + 3x_3 \le 2{,}400$$

$$x_1 + 3x_2 + 3x_3 \le 4{,}800$$

and $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.$

Example 20. *A firm buys castings of P and Q type of parts and sells them as finished product after machining, boring and polishing. The purchasing cost for castings are Rs. 3 and Rs. 4 each for parts P and Q and selling costs are Rs. 8 and Rs. 10 respectively. The per hour capacity of machines used for machining, boring and polishing for two products is given below :*

Capacity per hour	Parts	
	P	Q
Machining	30	50
Boring	30	45
Polishing	45	30

The running costs for machining, boring and polishing are Rs. 30, Rs. 22·5 and Rs. 22·5 per hour respectively. FORMULATE the linear programming problem to find out the product mix to maximize the profit.

[C.A. Nov. 97]

Solution. Let x_1, x_2 be the number of P and Q type parts to be produced per hour respectively.

For the profit part of x_1 and x_2, we calculate the total cost for each and then subtract the sale price of that part from it. The cost and profit per part are calculated in the following table :

Casting operation	x_1	x_2
Machining	30/30 = 1·00	30/50 = 0·60
Boring	22·5/30 = 0·75	22·5/45 = 0·50
Polishing	22·5/45 = 0·50	22·5/30 = 0·75
Purchase	3	4
Total cost	5·?5	5·85
Sale price	8	10
Profit	2·75	4·15

On the machine, type of P parts consumes 1/30th of the available hour, a type Q part consumes 1/50th of an hour. Thus, the machine constraint becomes :

$$\frac{1}{30}x_1 + \frac{1}{50}x_2 \leq 1 \quad \text{or} \quad 50\,x_1 + 30\,x_2 \leq 1{,}500$$

Similarly, other constraints can be established.

The linear programming formulation of the given problem will be :

$$\textbf{Max. (Total profit)}\ P = 2{\cdot}75\,x_1 + 4{\cdot}15\,x_2,$$

subject to the constraints :

$$\frac{1}{30}x_1 + \frac{1}{50}x_2 \leq 1 \text{ or } 50\,x_1 + 30\,x_2 \leq 1{,}500 \text{ (machine constraint)}$$

$$\frac{1}{30}x_1 + \frac{1}{45}x_2 \leq 1 \text{ or } 45\,x_1 + 30\,x_2 \leq 1{,}350 \text{ (boring constraint)}$$

$$\frac{1}{45}x_1 + \frac{1}{30}x_2 \leq 1 \text{ or } 30\,x_1 + 45\,x_2 \leq 1{,}350 \text{ (polishing constraint)}$$

and $$x_1, x_2 \geq 0.$$

Example 21. *The owner of Fancy Goods shop is interested to determine how many advertisements to release in the selected three magazines, A, B* and *C. His main purpose is to advertise in such a way that total exposure to principal buyers of his goods is maximised. Percentages of readers for each magazine are known. Exposure in any particular magazine is the number of advertisements released multiplied by the number of principal buyers. The following data are available :*

Particulars	*Magazines*		
	A	B	C
Readers	1·0 lakh	0·6 lakh	0·4 lakh
Principal buyers	20%	15%	8%
Cost per Adv.	8,000	6,000	5,000

The budgeted amount is at the most Rs. 1·0 lakh for the advertisements. The owner has already decided that magazine A should have no more than 15 advertisements and that B and *C each gets at least 8 advertisements. FORMULATE a linear programming model for this problem. DO NOT SOLVE.* **[C.A. Nov. 96]**

Solution. Let x_1, x_2 and x_3 be the required number of insertions in magazine A, B and C respectively. The total exposure of principal buyers of the magazine is

$$Z = (20\% \text{ of } 1{,}00{,}000)\,x_1 + (15\% \text{ of } 60{,}000)\,x_2 + (8\% \text{ of } 40{,}000)\,x_3$$

$$8{,}000\,x_1 + 6{,}000\,x_2 + 5{,}000\,x_3 \le 1{,}00{,}000 \qquad \textit{(budgeting constraint)}$$
$$x_1 \le 15,\ x_2 \ge 8 \text{ and } x_3 \ge 8 \qquad \textit{(advertisement constraint)}$$

The LP model is :

Max. (Total exposure) $P = 20{,}000\,x_1 + 9{,}000\,x_2 + 3{,}200\,x_3$,

subject to the constraints :

$$8{,}000\,x_1 + 6{,}000\,x_2 + 5{,}000\,x_3 \le 1{,}00{,}000$$
$$0 \le x_1 \le 15,\ 0 \le x_2 \le 8,\ 0 \ge x_3 \ge 8.$$

Example 22. *Evening shift resident doctors in a Govt. hospital work five consecutive days and have two consecutive days off. Their five days of work can start on any day of the week and the schedule rotates indefinitely. The hospital requires the following minimum number of doctors working :*

Sun.	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
35	55	60	50	60	50	45

No more than 40 doctors can start their five working days on the same day. Formulate the general linear programming model to minimize the number of doctors employed by the hospital. **[Delhi (MBA) 98]**

Solution. Let $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ be the number of doctors who start their duty on j^{th} $(j = 1, 2, \dots, 7)$ day of the week. The given problem has the following LP formulation :

Max. (Total no. of doctors) $P = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$,

subject to the constraints :

$$x_1 + x_4 + x_5 + x_6 + x_7 \ge 35$$
$$x_2 + x_5 + x_6 + x_7 + x_1 \ge 55$$
$$x_3 + x_6 + x_7 + x_1 + x_2 \ge 60$$
$$x_4 + x_7 + x_1 + x_2 + x_3 \ge 50$$
$$x_5 + x_1 + x_2 + x_3 + x_4 \ge 60$$
$$x_6 + x_2 + x_3 + x_4 + x_5 \ge 50$$
$$x_7 + x_3 + x_4 + x_5 + x_6 \ge 45$$
$$0 \le x_j \le 40\ (j = 1, 2, \dots, 7).$$

Example 23. *The Omega Data Processing Company performs three types of activity : pay rolls, account receivables, and inventories. The profit and time requirements for key punch computation and office printing for a standard job are shown in the following table :*

Omega guarantees over night completion of the job. Any job schedule during the day can be completed during the day or night. Any job scheduled during the night, however, must be completed during the night. The capacity for both day and night are shown in the following table :

Capacity (Min.)	*Key punch*	*Computation*	*Print*
Day	4,200	150	400
Night	9,200	250	650

Formulate the linear programming problem in order to determine the 'mixture' of standard jobs that should be accepted during the day and night.

Solution. Let x_{ij} represent the jobs accepted during day and night.

The LP model is :

$$\text{Max. } P = 275\,(x_{11} + x_{12}) + 125\,(x_{21} + x_{22}) + 225\,(x_{31} + x_{32}),$$

subject to the constraints :

$$1200\,(x_{11} + x_{12}) + 1400\,(x_{21} + x_{22}) + 800\,(x_{31} + x_{32}) \ge 13{,}400$$
$$100\,(x_{11} + x_{12}) + 60\,(x_{21} + x_{22}) + 80\,(x_{31} + x_{32}) \le 1{,}050$$
$$1{,}200\,x_{12} + 1{,}400\,x_{22} + 800\,x_{32} \le 9{,}200$$
$$100\,x_{12} + 60x_{22} + 80\,x_{32} \le 650$$

and $\quad x_{ij} \ge 0$ for all i and j.

Example 24. *PQR Coffee Company mixes South Indian, Assamese and Imported coffee to make two brands of coffee, Plains X and Plains XX. The characteristics used in blending the coffees include strength, acidity and caffeine. The test results of the available supplies of South Indian, Assamese, and Imported coffees are shown in the following table :*

	Price/kg (Rs.)	*Strength index*	*Acidity index*	*Percent caffeine*	*Supply available* (kgs.)
South Indian	30	6	4·0	2·0	40,000
Assamese	40	8	3·0	2·5	20,000
Imported	35	5	3·5	1·5	15,000

The requirements for Plains X and Plains XX coffees are given in the following table :

Plains Coffee	*Price/kg (Rs.)*	*Min. Strength*	*Max. Acidity*	*Max. per cent caffeine*	*Quantity Demanded* (kgs)
X	45	6·5	3·8	2·2	35,000
XX	55	6·0	3·5	2·0	25,000

Assume that 35,000 kgs of Plains X and 25,000 kgs. of Plains XX are profits to be sold. Formulate the LPP to maximize profit. **[Bombay (MMS) 96]**

Solution. For Plain Coffee *X*, let

x_{11} = quantity in kg of South Indian coffee

x_{12} = quantity in kg of Assamese coffee

x_{13} = quantity in kg of Imported coffee

and for Plain Coffee *XX*

x_{21} = quantity in kg of South Indian coffee

x_{22} = quantity in kg of Assamese coffee

x_{23} = quantity in kg of Imported coffee

Thus, the LP model can be presented as follows :

$$\text{Max. (Profit) } P = 45\,(30x_{11} + 40x_{12} + 35x_{13}) + 55\,(30\,x_{21} + 40\,x_{22} + 35\,x_{23}),$$

subject to the constraints

$$\left.\begin{aligned} 6x_{11} + 8x_{12} + 5x_{13} &\geq 6{\cdot}5 \\ 6x_{21} + 8x_{22} + 5x_{23} &\geq 6{\cdot}0 \end{aligned}\right\} \textit{strength index constraint}$$

$$\left.\begin{aligned} 4x_{11} + 3x_{12} + 3{\cdot}5\,x_{13} &\leq 3{\cdot}8 \\ 4x_{21} + 3x_{22} + 3{\cdot}5\,x_{23} &\leq 3{\cdot}5 \end{aligned}\right\} \textit{acidity constraint}$$

$$\left.\begin{aligned} 2x_{11} + 2{\cdot}5\,x_{12} + 1{\cdot}5\,x_{13} &\leq 2{\cdot}2 \\ 2x_{21} + 2{\cdot}5\,x_{22} + 2{\cdot}5\,x_{23} &\leq 2{\cdot}0 \end{aligned}\right\} \textit{caffeine constraint}$$

$$\left.\begin{aligned} x_{11} + x_{21} &\leq 40{,}000 \\ x_{12} + x_{22} &\leq 20{,}000 \\ x_{13} + x_{23} &\leq 15{,}000 \end{aligned}\right\} \textit{sale constraint}$$

$$\left.\begin{aligned} x_{11} + x_{12} + x_{13} &= 35{,}000 \\ x_{21} + x_{22} + x_{23} &= 25{,}000 \end{aligned}\right\} \textit{sale constraint}$$

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0.$$

Example 25. *A manufacturer of biscuits is considering four types of gift packs containing three types of biscuits : orange cream (OC), chocolate cream (CC), and Wafors (W). Market research study conducted recently to assess the preferences of the consumers shows the following types of assortments to be in good demand :*

Assortments	*Contents*	*Selling price per kg.*
A	Not less than 40% of OC Not more than 20% of CC Any quantity of W	20
B	Not less than 20% of OC Not more than 40% of CC Any quantity of W	25
C	Not less than 50% of OC Not more than 10% of CC Any quantity of W	22
D	No restrictions	12

For the biscuits, the manufacturing capacity and costs are given below :

Biscuits variety	*Plant capacity* (kg/day)	*Manufacturing cost* Rs/kg.
OC	200	8
CC	200	9
W	150	7

Formulate a linear programming model to find the production schedule which maximizes the profit assuming that there are no market restrictions. **[Delhi (MBA) April, 99]**

(Note. This is also discussed earlier in Ex. 11 on page 61.)

Solution. Let the decision variables x_{ij} $(i = A, B, C, D; j = 1, 2, 3)$ be defined as follows :

(i) x_{A1}, x_{A2}, x_{A3}, denote the quantity in kg of OC, CC and W type of biscuits respectively for A.

(ii) x_{B1}, x_{B2}, x_{B3}, denote the quantity in kg of OC, OC and W type of biscuits, respectively, for B.

(iii) x_{C1}, x_{C2}, x_{C3}, denote the quantity in kg. of OC, CC and W type of biscuits respectively, for C.

(iv) x_{D1}, x_{D2}, x_{D3} denote the quantity in kg of OC, CC and W type of biscuits, respectively, for D.

The LP model can be easily obtained as :

$$\text{Max. } Z = 20\,(x_{A1} + x_{A2} + x_{A3}) + 25\,(x_{B1} + x_{B2} + x_{B3}) + 22\,(x_{C1} + x_{C2} + x_{C3})$$
$$+ 12\,(x_{D1} + x_{D2} + x_{D3}) - 8\,(x_{A1} + x_{B1} + x_{C1} + x_{D1}) - 9\,(x_{A2} + x_{B2} + x_{C2} + x_{D2}) - 7\,(x_{A3} + x_{B3} + x_{C3} + x_{D3})$$
$$= 12x_{A1} + 11x_{A2} + 13x_{A3} + 17x_{B1} + 16x_{B2} + 18x_{B3} + 24x_{C1} + 13x_{C2} + 15x_{C3} + 4x_{D1} + 3x_{D2} + 5x_{D3}$$

subject to the constraints :

$$\left.\begin{aligned} x_{A1} &\geq 0{\cdot}40\,(x_{A1} + x_{A2} + x_{A3}) \\ x_{A2} &\leq 0{\cdot}20\,(x_{A1} + x_{A2} + x_{A3}) \end{aligned}\right\}\text{(constraints for } A)$$
$$\left.\begin{aligned} x_{B1} &\geq 0{\cdot}20\,(x_{B1} + x_{B2} + x_{B3}) \\ x_{B2} &\leq 0{\cdot}40\,(x_{B1} + x_{B2} + x_{B3}) \end{aligned}\right\}\text{(constraints for } B)$$
$$\left.\begin{aligned} x_{C1} &\geq 0{\cdot}50\,(x_{C1} + x_{C2} + x_{C3}) \\ x_{C2} &\leq 0{\cdot}20\,(x_{C1} + x_{C2} + x_{C3}) \end{aligned}\right\}\text{(constraints for } C)$$
$$\left.\begin{aligned} x_{A1} + x_{B1} + x_{C1} + x_{D1} &\leq 200 \\ x_{A2} + x_{B2} + x_{C2} + x_{D2} &\leq 200 \\ x_{A3} + x_{B3} + x_{C3} + x_{D3} &\leq 150 \end{aligned}\right\}\text{(plant capacity constraints)}$$
$$x_{ij} \geq 0 \;\; (i = A, B, C, D;\; j = 1, 2, 3)$$

EXAMINATION PROBLEMS

1. A firm manufactures headache pills in two sizes A and B. Size A contains 2 grains of aspirin, 5 grains of bicarbonate and 1 grain of codeine. Size B contains 1 grain of aspirin, 8 grains of bicarbonate and 6 grains of codeine. It is found by users that it requires at least 12 grains of aspirin, 74 grains of bicarbonate and 24 grains of codeine for providing immediate effect. It is required to determine the least number of pills a patient should take to get immediate relief. Formulate the problem as a standard LPP. **[Kota 91]**

   [Ans. Min. $z = x_1 + x_2$, *s.t.* $2x_1 + x_2 \geq 12$, $5x_1 + 8x_2 \geq 74$, $x_1 + 6x_2 \geq 24$; $x_1, x_2 \geq 0$.]

2. A manufacturer has three machines A, B, C with which he produces three different articles P, Q, R. The different machine times required per article, the amount of time available in any week on each machine and the estimated profits per article are furnished in the following table :

Article	*Machine time (in hrs.)*			*Profit per article (in rupees)*
	A	B	C	
P	8	4	2	20
Q	2	3	0	6
R	3	0	1	8
Available machine hrs.	250	150	50	

Formulate the problem as a linear programming problem.

[**Ans.** Max. $z = 20P + 6Q + 8R$; such that $8P + 2Q + 3R \le 250$, $4P + 3Q \le 150$, $2P + R \le 50$; and $P, Q, R \ge 0$.]

3. Two alloys A and B are made from four different metals I, II, III and IV according to the following specifications:

A : at most 80% of I, at most 30% of II, at least 50% of III, B : between 40% & 60% of II, at least 30% of III, at most 70% of IV.

The four metals are extracted from three different ores whose consituents percentage of these metals, maximum available quantity and cost per ton are as follows :

Constituent %

Ore	*Max. Quantity (tons)*	I	II	III	IV	Others	Price (Rs. per ton)
1	1000	20	10	30	30	10	30
2	2000	10	20	30	30	10	40
3	3000	5	5	70	20	0	50

Assuming the selling prices of alloys A and B are Rs. 200 and Rs. 300 per ton respectively. Formulate the above as a linear programming problem selecting appopriate objective and constraint functions. **[I.C.W.A. (Dec.) 88]**

[**Ans.** Max. $z = 200A + 300B - 30\,(x_{1A} + x_{2B}) - 40\,(x_{2A} + x_{2B}) - 50\,(x_{3A} + x_{3B})$, subject to the constraints :

$$\left.\begin{aligned} 0{\cdot}20x_{1A} + 0{\cdot}10x_{2A} + 0{\cdot}05x_{3A} &\le 0{\cdot}80A \\ 0{\cdot}10x_{1A} + 0{\cdot}20x_{2A} + 0{\cdot}05x_{3A} &\le 0{\cdot}30A \\ 0{\cdot}30x_{1A} + 0{\cdot}30x_{2A} + 0{\cdot}70x_{3A} &\ge 0{\cdot}50A \end{aligned}\right\}\text{(alloy specification for } A) \quad \left.\begin{aligned} 0{\cdot}1x_{1B} + 0{\cdot}2x_{2B} + 0{\cdot}05x_{3B} &\ge 0{\cdot}40B \\ 0{\cdot}1x_{1B} + 0{\cdot}2x_{2B} + 0{\cdot}05x_{3B} &\le 0{\cdot}6B \\ 0{\cdot}3x_{1B} + 0{\cdot}3x_{2B} + 0{\cdot}7x_{3B} &\ge 0{\cdot}3B \\ 0{\cdot}3x_{1B} + 0{\cdot}3x_{2B} + 0{\cdot}2x_{3B} &\le 0{\cdot}7B \end{aligned}\right\}\text{(alloy specification for } B)$$

$$\left.\begin{aligned} 0{\cdot}6x_{1A} + 0{\cdot}6x_{2A} + 0{\cdot}3x_{3A} &= A \\ 0{\cdot}7x_{1B} + 0{\cdot}8x_{2B} + 0{\cdot}95x_{3B} &= B \end{aligned}\right\}\text{(material balance)} \quad \left.\begin{aligned} x_{1A} + x_{1B} &\le 1000 \\ x_{2A} + x_{2B} &\le 2000 \\ x_{3A} + x_{3B} &\le 3000 \end{aligned}\right\}\text{(availability of ores)}$$

$A, B, x_{iA}, x_{iB} \ge 0$ $(i = 1, 2, 3)$, where A and B denote quantities of alloy of A and B; and x_{iA}, x_{iB} $(i = 1, 2, 3)$ denote the quantities going into A or B from I, II, III, or IV.]

4. Consider the following problem faced by a production planner in a soft drink plant. He has two bottling machines A and B. A is designed for 8-ounce bottles and B for 16-ounce bottles. However, each can be used on both types with some loss of efficiency. The following data is available.

Machine	*8-ounce bottles*	*16-ounce bottles*
A	100/minute	40/minute
B	60/minute	75/minute

The machine can be run 8-hour per day, 5 days per week. profit on 8-ounce bottle is 15 paise and on 16-ounce is 25 paise. Weekly production of the drink cannot exceed 3,00,000 ounces and the market can absorb 25,000 eight-ounce bottles and 7,000 sixteen-ounce bottles per week. The planner wishes to maximize his profit subject, of course, to all the production and marketing restrictions, formulate this as linear programming problem.

[**Hint.** Let x_1 units of 8-ounce bottle and x_2 units of 16-ounce bottle be produced. Total profit of production planner is given by $P = 0.15x_1 + 0.25x_2$.

Since machine A and B both work 8 hours per day and 5 days per week, the total working time for machine A and B will become 2400 minutes per week. Therefore, time conditions will be

$$\frac{x_1}{100} + \frac{x_2}{40} \le 2400 \text{ (for machine } A), \quad \frac{x_1}{60} + \frac{x_2}{75} \le 400 \text{ (for machine } B)$$

Restriction of total weekly production will be $8x_1 + 16x_2 \le 3{,}00{,}000$.

Market consumption is restricted by $0 \le x_1 \le 25000$ and $0 \le x_2 \le 7000$.

5. Formulate the following linear programming problem.

A used-car dealer wishes to stock-up his lot to maximize his profit. He can select cars A, B and C which are valued wholesale at Rs. 5000, Rs. 7000 and Rs. 8000 respectively. These can be sold at Rs. 6000, 8500 and 10500 respectively.

For each car type, the probabilities of sale are :

Type of car	:	A	B	C
Prob. of sale in 90 days.	:	0·7	0·8	0·6

For every two cars of B, he should buy one car of type A or C. If he has Rs, 1,00,000 to invest, what should he buy to maximize his expected gain. **[VTU (BE VIth Sem.) Aug. 2002]**

[**Hint.** Let x_1, x_2, x_3 number of cars be purchased of type A, B, C respectively. Gain per car for A, B, C will be Rs. (6000 – 50000), Rs. (85000 – 7000), Rs. (10500 – 8000) respectively. Therefore total expected gain will be

$$z = 1000\, x_1 \times 0{\cdot}7 + 1500 x_2\, 0{\cdot}8 + 2500 x_3 \times 0{\cdot}6]$$

Investment constraints will be given by

$$5000x_1 + 7000\,(2x_2) \le 1{,}00{,}000 \text{ and } 7000\,(2x_2) + 8000\, x_3 \le 1{,}00{,}000].$$

6. Certain farming organization operated three farms of comparable productivity. The output of each farm is limited both by the usable acreage and by the amount of water available for irrigation. The data for the upcoming season are the following :

Farm	*Usable acreage*	*Water available in acre feet*
1	400	1500
2	600	2000
3	300	900

The organization is considering three crops for planting which differ primarily in their expected profit per acre and in their consumption of water. Furthermore, the total acreage that can be devoted to each of the crops is limited by the amount of appropriate harvesting equipment available.

Crop	**Minimum acreage**	**Water consumption in acre feet**	**Expected profit per acre**
A	700	5	Rs. 400
B	800	4	Rs. 300
C	300	3	Rs. 100

In order to maintain a uniform work-load among the farms, it is the policy of the organization that the percentage of the usable acreage planted must be the same at each farm. However, any combination of the crops may be grown at any of the farms. The organization wishes to know how much of each crop should be planted at the respective farms in order to maximize expected profit. Formulate this as a linear programming problem. **[Roorkee (B.E. IV th) 91; Delhi (M.B.A.) 75]**

[**Hint.** Let the number of acres at the ith farm devoted to jth crop be denoted by the decision variable x_{ij} $(i = 1, 2, 3\ ;\ j = A, B, C)$.

Then the formulation of the problem will be :

$$\text{Max. } \mathbf{P} = 400 \sum_{i=1}^{3} x_{iA} + 300 \sum_{i=1}^{3} x_{iB} + 100 \sum_{i=1}^{3} x_{iC}, \text{ subject to, } \sum_{j=A}^{C} x_{1j} \le 400,\ \sum_{j=A}^{C} x_{2j} \le 600,\ \sum_{j=A}^{C} x_{3j} \le 300.$$

$$\left\{\begin{array}{l} 5x_{iA} + 4x_{iB} + 3x_{iC} \le 1{,}5000 \\ 5x_{2A} + 4x_{2B} + 3x_{2C} \le 2{,}000 \\ 5x_{3A} + 4x_{3B} + 3x_{3C} \le 900 \end{array}\right\},\quad \sum_{j=A}^{C} x_{1j} \le 700,\ \sum_{j=A}^{C} x_{2j} \le 800,\ \sum_{j=A}^{C} x_{3j} \le 300,$$

As the farming organization wishes to keep the policy of uniform workload, the following equations must also hold :

$$\frac{x_{1A} + x_{1B} + x_{1C}}{400} = \frac{x_{2A} + x_{2B} + x_{2C}}{600},\ \frac{x_{2A} + x_{2B} + x_{2C}}{600} = \frac{x_{3A} + x_{3B} + x_{3C}}{300},\ \frac{x_{1A} + x_{1B} + x_{1C}}{400} = \frac{x_{3A} + x_{3B} + x_{3C}}{300}.$$

Since the first two equations yield the third, the third equation can be omitted from the model. Now rearranging the remaining two equations, the uniform workload restrctions become :

$$3\,(x_{1A} + x_{1B} + x_{1C}) - 2\,(x_{2A} + x_{2B} + x_{2C}) = 0 \text{ and } (x_{2A} + x_{2B} + x_{2C}) - 2\,(x_{3A} + x_{3B} + x_{3C}) = 0.$$

7. A feed mixing company purchases and mixes one or more of the three types of grain, each containing different amount of three nutritional elements, the data is given below :

Item Nutritional ingredient	*One unit weight of*			*Minimum total requirement, over planning horizon*
	Grain 1	*Grain 2*	*Grain 3*	
A	2	4	6	≥ 125
B	0	2	5	≥ 24
C	5	1	3	≥ 80
Cost per unit wt. (Rs.)	25	15	18	Minimize

The production manager specifies that any feed mix for his livestock must meet at least minimal nutritional requirements; and seeks the least costly among all such mixes. Suppose his planning horizon is a two week period, that is, he purchases enough to fill his needs for two weeks. Formulate this as an L.P.P.

[**Hint.** Let x_1, x_2, x_3 denote the weight levels of three different grains. Then by considering the nutritional ingredient in the three grains, linear programming problem is :

To minimize $\mathbf{C} = 25x_1 + 15x_2 + 18x_3$, subject to the constraints :

$$2x_1 + 4x_2 + 6x_3 \ge 125,\ 2x_2 + 5x_3 \ge 24,\ 5x_1 + x_2 + 3x_3 \ge 80, \text{ and } x_1, x_2, x_3 \ge 0]$$

8. *ABC* foods company is developing a low calorie high protein diet supplement called *Hi*-Pro. The specification for *Hi*-Pro, have been established by a panel of medical experts. These specifications along with the calorie, protein and vitamin content of three basic foods are given in the following table :

Units of Nutritional Elements per 100 gm Serving of Basic Foods :

Nutritional Elements	Basic Foods			Hi-Pro. Specifications
	No. 1	No. 2	No. 3	
Calories	350	250	200	≤ 300
Protein	250	300	150	≥ 200
Vitamin A	100	150	75	≥ 100
Vitamin C	75	125	150	≥ 100
Cost per serving (Rs.)	1.50	2·00	1·20	

Formulate the linear programming model to minimize cost.

[Hint. Let x_1, x_2, x_3 denote the number of units of basic food number 1, 2 and 3, respectively. Then the formulation will be obtained as follows : Min. $\mathbf{z} = 1{\cdot}50x_1 + 2x_2 + 1{\cdot}20x_3$, subject to the conditions :

$350x_1 + 250x_2 + 200x_3 \le 300$ $\quad$ $100x_1 + 150x_2 + 75x_3 \ge 100$ $\quad$ $x_1, x_2, x_3 \ge 0$.]

$250x_1 + 300x_2 + 150x_3 \ge 200$ $\quad$ $75x_1 + 125x_2 + 150x_{23} \ge 100$

9. **(Investment Problem)** Mr. Krishnamutry, a retired Govt. officer, has recently received his retirement benefits, *viz.*, provident fund, gratuity, etc. He is contemplating as to how much funds he should invest in various alternatives open to him so as to maximize return on investment. The investment alternatives are : government securities, fixed deposits of a public limited company, equity shares, time deposits in banks, national saving certificates and real estate. He has made a subjective estimate of the risk involved on a five point scale. The data on the return on investment, the number of years for which the funds will be blocked to earn this return on investment and the subjective risk involved are as follows :

Investment Alternates	**Return**	**Number of years**	**Risk**
Govt. securities	6%	15	1
Company deoposits	15%	3	3
Equity shares	20%	6	7
Time deposits	10%	3	1
NSC	12%	6	1
Real Estate	25%	10	2

He was wondering what percentage of funds he should invest in each alternative so as to maximize the return on investment. He decided that average risk should not be more than 4, and funds should not be locked up for more than 15 years. He would necessarily invest at least 30% in real estate.

Formulate an LP model for the problem.

[Hint Let x_1, x_2, x_3, x_4 and x_5 be the percentage of the total funds that should be invested in government securities, company deposits, equity shares, time deposits, national saving certificates and real estate, respectively. The LP model can be formulated as follows :

Maximize $\mathbf{z} = 0{\cdot}06x_1 + 0{\cdot}15x_2 + 0{\cdot}20x_3 + 0{\cdot}10x_4 + + 0{\cdot}12x_5 + 0{\cdot}25x_6$, subject to the conditions :

$15x_1 + 3x_2 + 6x_3 + 3x_4 + 6x_5 + 10x_6 \le 15$, $x_1 + 3x_2 + 7x_3 + x_4 + x_5 + 2x_6 \le 4$, $x_1, x_2, x_3 \ge 0$.

$x_6 \ge 0{\cdot}30$; $x_1, x_2, x_3, x_4, x_5, x_6 \ge 0$] .

10. **(Media Selection Problem)** The Owner of Metro Sports wishes to determine how many advertisements to place in the selected three monthly magazines *A*, *B* and *C*. His objective is to advertise in such a way that total exposure to principal buyers of expensive sports goods is maximized. Percentages of readers for each magazine are known, Exposure in any particular magazine is the number of advertisments placed multiplied by the number of principal buyers. The following data may be used :

Exposure Category	Magazines		
	A	B	C
Readers (in Lakhs)	1	0·6	0·4
Principal Buyers	10%	15%	7%
Cost per advertisement (Rs.)	5000	4500	4250

The budgeted amount is at most Rs. 1 Lakh for the advertisements. The owner has already decided that magazine *A* should have no more than 6 advertisements and that *B* and *C* each have at least two advertisements. Formulate a LP model for the problem.

[Hint. Let x_1, x_2, x_3 be the number of insertions in magazine *A*, *B* and *C*, respectively.

Then the LP formulation will be as follows :

Maximize : $\mathbf{z}$ = (10% of 1,00,000) x_1 + (15% of 60,000) x_2 + (7% of 40,000) x_3 (Total exposure)

subject to the constraints : $5000\,x_1 + 4500\,x_2 + 4250\,x_3 \le 1{,}00{,}000$ (Budgeting constraint)

$x_1 \le 6$, $x_2 \ge 2$, $x_3 \ge 2$ (Advertisements constraint) $x_1 \ge 0$, $x_2 \ge 0$, $x_3 \ge 0$.]

11. A city hospital has the following minimal daily requirements for nurses.

Period	Clock Time (24 hr. day)	Minimal Number of Nurses Required
1	6 A.M. – 10 A.M.	2
2	10 A.M. – 2 P. M.	7
3	2 P.M. – 6 P.M.	15
4	6 P.M. – 10 P.M.	8
5	10 P.M. – 2 A.M.	20
6	2 A.M. – 6 A.M.	6

Nurses report to the hospital at the beginning of each period and work for 8 consecutive hours. The hospital wants to determine the minimum number of nurses to be employed so that there will be sufficient number of nurses available for each period. Formulate this as a linear programming problem by setting up appropriate constraints and objective function. Do not solve.

[**Hint.** Let $x_1, x_2, x_3, x_4, x_5, x_6$ be the number of nurses on duty at 6 *A.M.*, 10 *A.M.*, 2 *P.M.size5* , 6 *P.M.*, 10 *P.M.*, and 2 *A.M.*, respectively. Then the required LP formulation will be as follows :

Minimize $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$

subject to the constraints :

$$x_1 + x_2 \geq 7$$
$$x_2 + x_3 \geq 15$$
$$x_3 + x_4 \geq 8$$
$$x_4 + x_5 \geq 20$$
$$x_5 + x_6 \geq 6$$
$$x_6 + x_1 \geq 2$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0].$$

Graphical Method

3·3. GRAPHICAL SOLUTION OF TWO VARIABLE PROBLEMS

3.3–1. Graphical Procedure

Simple linear programming problems of two decision variables can be easily solved by *graphical method.*

The outlines of graphical procedure are as follows :

Step 1. Consider each inequality–constraint as equation.

Step 2. Plot each equation on the graph, as each one will geometrically represent a straight line.

Step 3. Shade the feasible region. Every point on the line will satisfy the equation of the line. If the inequality-constraint corresponding to that line is ' ≤ ', then the region *below* the line *lying* in the first quadrant (due to non-negativity of variables) is shaded. For the inequality-constraint with ' ≥ ' sign, the region *above* the line in the first quadrant is shaded. The points lying in common region will satisfy all the constraints simultaneously. The common region thus obtained is called the ***feasible region.***

Step 4. Choose the convenient value of **z** (say = 0) and plot the objective function line.

Step 5. Pull the objective function line until the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin and passing through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin and passing through at least one corner of the feasible region.

Step 6. Read the coordinates of the extreme point(s) selected in *Step 5*, and find the maximum or mininum (as the case may be) value of **z**. The following examples will make the outlined graphical procedure clear.

Q. 1. What is meant by linear programming problem ? Give brief description of the problem with illustrations. How the same can be solved graphically. What are the basic characteristics of a linear programming problem ? **[Meerut (Stat.) 98]**

2. Explain briefly the graphical method of solving linear programming problems. State its advantages and limitations.

3. Write the algorithm of graphical solution for LP models. **[JNTU (MCA III) 2004]**

3·3-2. Graphical Solution of Properly Behaved LP Problems

Example 26. *Find a geometrical interpretation and solution as well for the following LP problem :*

Maximize $\mathbf{z} = 3x_1 + 5x_2$, *subject to restrictions :*

$x_1 + 2x_2 \leq 2000,\ x_1 + x_2 \leq 1500, x_2 \leq 600,$ *and* $x_1 \geq 0, x_2 \geq 0$.

Graphical Solution.

Step 1. ***(To graph the inequality-constraints).*** Consider two mutually perpendicular lines OX_1 and OX_2 as axes of coordinates. Obviously, any point (x_1, x_2) in the positive quadrant will certainly satisfy non-negativity restrictions : $x_1 \geq 0, x_2 \geq 0$. To plot the line $x_1 + 2x_2 = 2000$, put $x_2 = 0$, find $x_1 = 2000$ from this equation.

Then mark a point L such that $OL = 2000$ by assuming a suitable scale, say 500 units = 2 cm. Similarly, again put $x_1 = 0$ to find $x_2 = 1000$ and mark another point M such that $OM = 1000$.

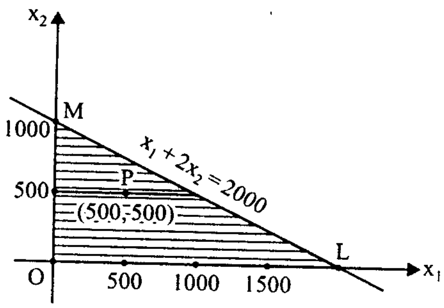


Fig. 3.1

Now join the points L and M. This line will represent the equation $x_1 + 2x_2 = 2000$ as shown in the above figure.

Clearly, any point P lying on or below the line $x_1 + 2x_2 = 2000$ will satisfy the inequality $x_1 + 2x_2 \leq 2000$. (If we take a point (500, 500), *i.e.*, $x_1 = 500$, $x_2 = 500$, then we have $500 + 2 \times 500 < 2000$, which is true).

Similar procedure is now adoped to plot the other two lines : $x_1 + x_2 = 1500$ and $x_2 = 600$ as shown in the Figs. 3.2 and 3.3, respectively. Any point on or below the lines $x_1 + x_2 = 1500$ and $x_2 = 600$ will also satisfy other two inequalities : $x_1 + x_2 \leq 1500$, and $x_2 \leq 600$, respectively.

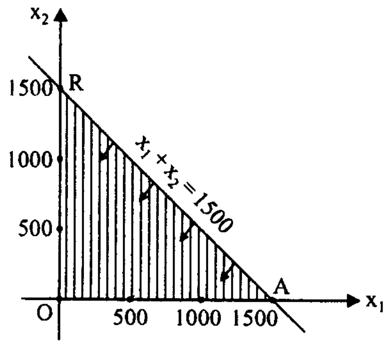


Fig. 3.2

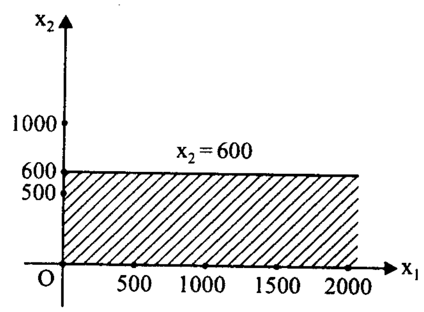


Fig. 3.3

Step 2. Find the *feasible region* or *solution space* by combining the Figs. 3.1, 3.2 and 3.3 together. A common shaded area *OABCD* is obtained (see Fig. 3.4) which is a set of points satisfying the inequality constraints :

$$x_1 + 2x_2 \leq 2000, x_1 + x_2 \leq 1500, x_2 \leq 600,$$

and non-negativity restrictions as $x_1 \geq 0, x_2 \geq 0$. *Hence any point in the shaded area* (including its boundary) is feasible solution to the given LPP.

Step 3. Find the co-ordinates of the corner points of feasible region O, A, B, C and D.

Step 4. Locate the corner point of optimal solution either by calculating the value of **z** for each corner point O, A, B, C, and D (or by adopting the following procedure).

Here, the problem is to find the point or points in the feasible region (collection of all feasible solutions) which maximize(s) the objective (or profit) function. For some fixed value of **z**, $\mathbf{z} = 3x_1 + 5x_2$ is a straight line and *any point on it gives the same value of* **z**. Also, it should be noted that the lines corresponding to different values of **z** are parallel, because the gradient $(-3/5)$ of the line $\mathbf{z} = 3x_1 + 5x_2$ remains the same throughout. For $\mathbf{z} = 0$, *i.e.*, $0 = 3x_1 + 5x_2$, means a line which passes through the origin. To draw the line $3x_1 + 5x_2 = 0$, determine the ratio $\dfrac{x_1}{x_2} = \dfrac{-5}{3} = \dfrac{-500}{300}$.

Mark the point E moving 500 units distance from the origin on the negative side of x_1-axis. Then find the points F such that $EF = 300$ units in the positive direction of x_2-axis. Joining the point F and O, draw the line

$3x_1 + 5x_2 = 0$. Now go on drawing the lines parallel to this line until at least a line is found which is farthest from the origin but passes through at least one corner of the feasible region at which the maximum value of **z** is attained. It is also possible that such a line may coincide with one of the edge of feasible region. In that case, every point on that edge gives the maximum value of **z**.

In this example, maximum value of **z** is attained at the corner point B (1000, 500), which is the point of intersection of lines $x_1 + 2x_2 = 2000$ and $x_1 + x_2 = 1500$. Hence, the required solution is $x_1 = 1000$, $x_2 = 500$ and max. value **z** = Rs. 5500.

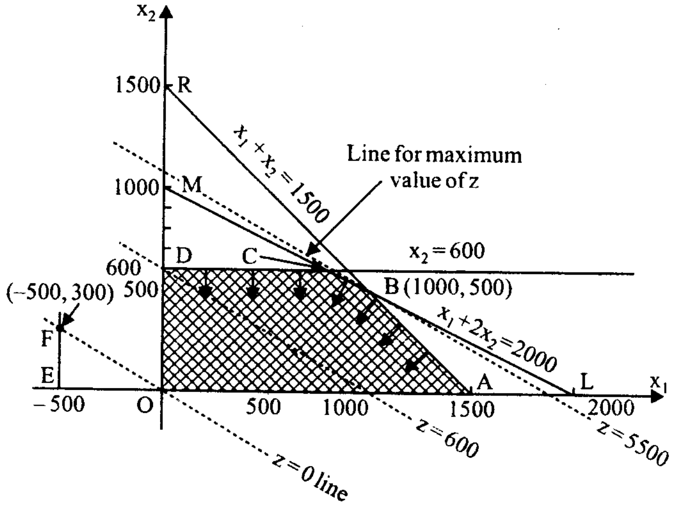


Fig. 3.4.

Note. If the number of vertices of feasible region is small, find the coordinates of vertices. As in above example, $O = (0, 0)$, $A = (1500, 0)$, $B = (1000, 500)$, $C = (800, 600)$, $D = (0, 600)$ are obtained by solving the pair of lines whose intersections are these points, respectively. The value of **z** corresponding to these points will be $z_O = 0$, $z_A = 4500$, $z_B = 5500$, $z_C = 4500$, $z_D = 3000$. Clearly $z_B = 5500$ is maximum for the point $B(1000, 500)$ which gives the required solution.

Example 27. *Consider the problem*

$$Max.\ \mathbf{z} = x_1 + x_2,\ subject\ to,$$

$$x_1 + 2x_2 \le 2000$$

$$x_1 + x_2 \le 1500$$

$$x_2 \le 600$$

and $$x_1, x_2 \ge 0.$$

Graphical Solution. This problem is of the same type as discussed earlier except the objective function is slightly changed. The feasible region will be similar to that of the above problem. Fig. 3.5 shows the objective function lines of the problem for three different values $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ of **z**.

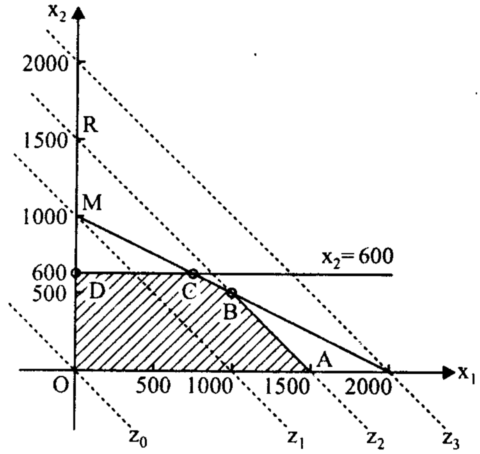


Fig. 3.5

It is clear from Fig. 3.5 that $\mathbf{z}_2$ is the maximum value of **z**. It is quite interesting that the line $\mathbf{z}_2$ representing the objective function lies along the edge AB of the polygon of feasible solutions. This indicates that the values of x_1 and x_2 which maximize **z** are not unique, but any point on the edge AB of $OABCD$ the polygon will give the optimum value of **z**. The maximum value of **z** is always unique, but there will be an infinite number of feasible solutions which give unique value of **z**. Thus, two corners A and B as well as any point on the line AB (segment) give optimal solution of this problem.

It should be noted that if a linear programming problem has more than one optimum solution, there exists alternative optimum solutions. And, one of the optimum solutions will be corresponding to corner point B, *i.e.* $x_1 = 1000, x_2 = 500$ with max. profit **z** = Rs. 1500.

Example 28. *Solve the following LP problem graphically :*

$$Max.\ z = 8000x_1 + 7000x_2,\ subject\ to$$

$$3x_1 + x_2 \le 66,\ x_1 + x_2 \le 45,\ x_1 \le 20,\ x_2 \le 40\ \ and\ x_1, x_2 \ge 0.$$

Solution. First, plot the lines $3x_1 + x_2 = 66$, $x_1 + x_2 = 45$, $x_1 = 20$ and $x_2 = 40$ and then shade the feasible region as shown in Fig. 3.6.

Draw a dotted line $8000x_1 + 7000x_2 = 0$ for $z = 0$ and continue to draw the lines till a point is obtained which is farthest from the origin but passing through at least one of the corners of the shaded (feasible) region. Fig. 3.6 shows that this point is $P(10{\cdot}5, 34{\cdot}5)$ which is the point of intersection of lines

$$3x_1 + x_2 = 66 \text{ and } x_1 + x_2 = 45.$$

Hence, z is maximum for $x_1 = 10{\cdot}5$ and $x_2 = 34{\cdot}5$

$$\text{Max. } z = 8000 \times 10{\cdot}5 + 7000 \times 34{\cdot}5$$
$$= \text{Rs. } 325000. \quad \textbf{Ans.}$$

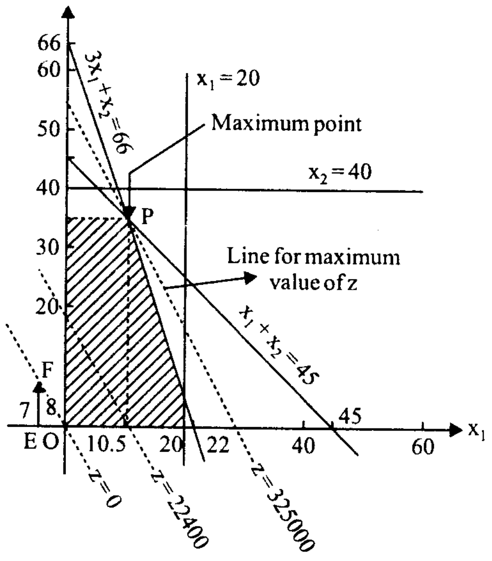


Fig. 3.6.

Example 29. *Old hens can be bought at Rs. 2 each and young ones at Rs. 5 each. The old hens lay 3 eggs per week and the young ones lay 5 eggs per week, each egg being worth 30 paise. A hen (young or old) costs Re. 1 per week to feed. I have only Rs. 80 to spend for hens, how many of each kind should I buy to give a profit of more than Rs. 6 per week, assuming that I cannot house more than 20 hens.*

[JNTU (B. Tech. 2002; Kota 93; Meerut (B.Sc.) 90, (M.Sc.) 90]

Solution. *Formulation.* Let x_1 be the number of old hens and x_2 the number of young hens to be bought.

Since old hens lay 3 eggs per week and the yound ones lay 5 eggs per week, the total number of eggs obtained per week will be $= 3x_1 + 5x_2$.

Consequently, the cost of each egg being 30 paise, the total gain will be= Rs. $0{\cdot}30\,(3x_1 + 5x_2)$.

Total expenditure for feeding $(x_1 + x_2)$ hens at the rate of Re. 1 each will be = Rs. $1 . (x_1 + x_2)$.

Thus, total profit **z** earned per week will be **z** = total gain − total expenditure

or $\quad \mathbf{z} = 0{\cdot}30\,(3x_1 + 5x_2) - (x_1 + x_2) \quad$ or $\quad \mathbf{z} = 0{\cdot}50\,x_2 - 0{\cdot}10\,x_1 \quad$ (objective

Since old hens can be bought at Rs. 2 each and young ones at Rs. 5 each and there are only Rs. 80 available for purchasing hens, the constraint is : $2x_1 + 5x_2 \le 80$.

Also, since it is not possible to house more then 20 hens at a time, $x_1 + x_2 \le 20$.

Also, since the profit is restricted to be more than Rs. 6, this means that the profit function **z** is to be maximized. Thus there is no need to add one more constraint, *i.e.* $0{\cdot}5x_2 - 0{\cdot}1\,x_1 \ge 6$.

Again, it is not possible to purchase negative quantity of hens, therefore $x_1 \ge 0$, $x_2 \ge 0$.

> Finally, the problem becomes :
> Find x_1 and x_2 (real numbers) so as to maximize the profit function $\mathbf{z} = 0{\cdot}50x_2 - 0{\cdot}10x_1$ subject to the constraints :
> $$2x_1 + 5x_2 \le 80,\ x_1 + x_2 \le 20, \text{ and } x_1, x_2 \ge 0.$$

Graphical Solution. Plot the straight lines $2x_1 + 5x_2 = 80$ and $x_1 + x_2 = 20$ on the graph and shade the feasible region as shown in the figure [Fig. 3.7].

The feasible region is $OBEC$. The coordinates of the extreme points of the feasible region are :

$O = (0, 0)$, $C = (20, 0)$, $B = (0, 16)$,
$E = (20/3, 40/3)$.

The values of z at these vertices are :

$z_o = 0$, $z_C = 0{\cdot}50 \times 0 - 0{\cdot}10 \times 20 = -2$,
$z_B = 0{\cdot}50 \times 16 - 0{\cdot}10 \times 0 = 8$,

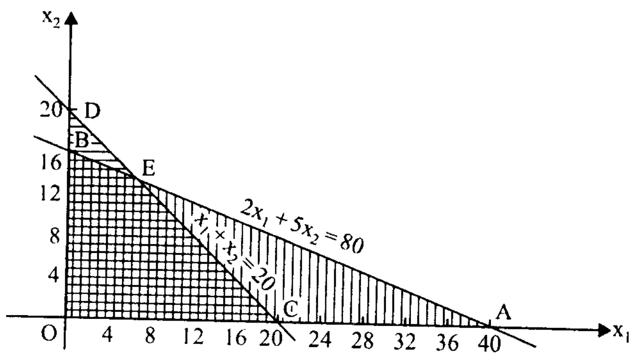


Fig. 3.7.

$$z_E = 0{\cdot}50 \times \frac{40}{3} - 0{\cdot}10 \times \frac{20}{3} = 6.$$

Since the maximum value of z is Rs. 8 which occurs at the point $B = (0, 16)$, the solution to the given problem is $x_1 = 0, x_2 = 16$, max. z = Rs. 8.

Hence only 16 young hens I should buy in order to get the maximum profit of Rs. 8 (which is > 6).

Example 30. ***(Minimization problem)***

Consider the problem : Min. $z = 1{\cdot}5x_1 + 2{\cdot}5x_2$ *subject to* $x_1 + 3x_2 \geq 3, x_1 + x_2 \geq 2, x_1, x_2 \geq 0$.

Graphical Solution. The geometrical interpretation of the problem is given in Fig. 3.8. The minimum value of z is $z_A = 3{\cdot}5$. This minimum is attained at the point of intersection A of the lines $x_1 + 3x_2 = 3$ and $x_1 + x_2 = 2$. This is the unique point to give the minimum value of z. Now, solving these two equations simultaneously, the optimum solution is : $x_1 = 3/2, x_2 = 1/2$ and min. $z = 3{\cdot}5$.

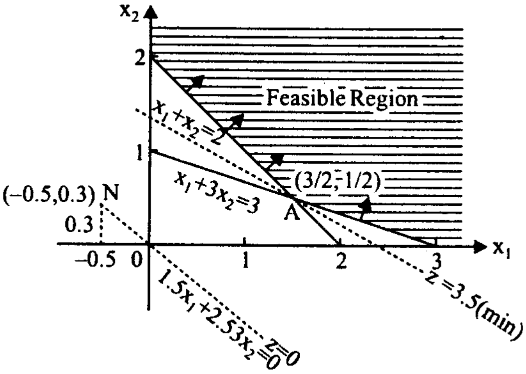


Fig. 3.8.

3·3–3. Graphical Solution in Some Exceptional Cases

The following examples show that there are certain exceptional cases which must be taken into consideration if a general technique for solving LP problems is to be developed.

Example 31. ***(Problem having unbounded solution)***

Max $z = 3x_1 + 2x_2$ *subject to* $x_1 - x_2 \leq 1, x_1 + x_2 \geq 3$, *and* $x_1, x_2 \geq 0$.

Graphical Solution. The region of feasible solutions is the shaded area in Fig. 3.9.

It is clear from this figure that the line representing the objective function can be moved far even parallel to itself in the direction of increasing z, and still have some points in the region of feasible solutions.

Hence z can be made arbitrarily large, and the problem has no finite maximum value of $\mathbf{z}$. Such problems are said to have *unbounded solutions.*

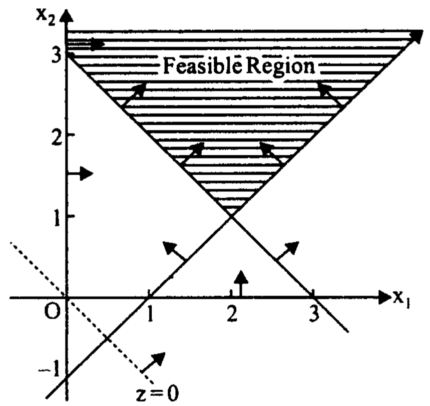


Fig. 3.9.

Infinite profit in practical problems of linear programming cannot be expected. If LP problem has been formulated by committing some mistake, it may lead to an unbounded solution.

Example 32. *Max.* $z = -3x_1 + 2x_2$ *subject to* $x_1 \leq 3$, $x_1 - x_2 \leq 0$, *and* $x_1, x_2 \geq 0$.

Graphical Solution. In *Example* 31. it has been seen that both the variables can be made arbitrarily large as z is increased. In this example, an unbounded solution does not necessarily imply that all the variables can be made arbitrarily large as z approaches infinity. Here the variable x_1 remains constant as shown in Fig. 3.10.

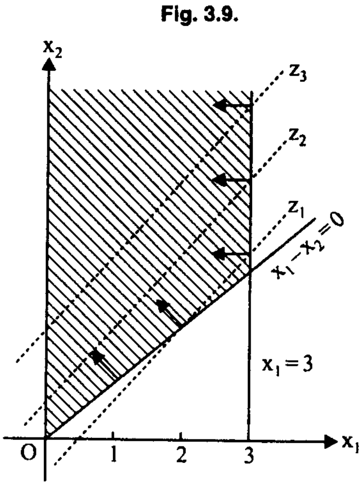


Fig. 3.10

Example 33. ***(Problem which is not completely normal)***

Maximize $z = -x_1 + 2x_2$ *subject to* $-x_1 + x_2 \leq 1$, $-x_1 + 2x_2 \leq 4$, *and* $x_1, x_2 \geq 0$.

Graphical Solution. The problem is solved graphically in Fig. 3.11.

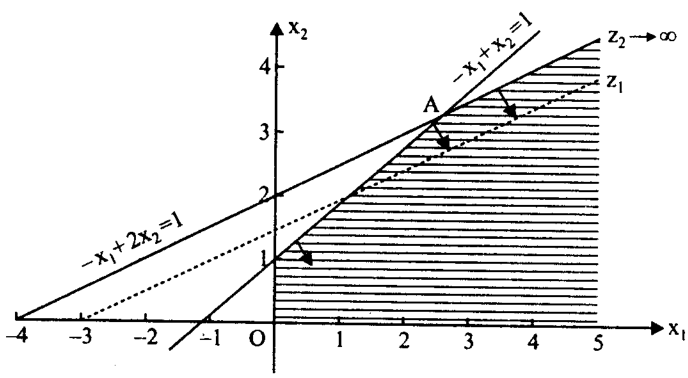


Fig. 3.11

Here the line of objective function coincides with the edge of Az_2 the region of feasible solutions. Thus, every point (x_1, x_2) lying on this edge $(-x_1 + 2x_2 = 4)$, which is going to infinity on the right gives $z = 4$, and is therefore an optimal solution.

Example 34. ***(Problem with inconsistent system of constraints)***

Maximize $z = 3x_1 - 2x_2$ *subject to* $x_1 + x_2 \leq 1, 2x_1 + 2x_2 \geq 4,$ *and* $x_1, x_2 \geq 0$.

Graphical Solution. The problem is represented graphically in Fig. 3.12.

The figure 3.12 shows that there is no point (x_1, x_2) which satisfies both the constraints simultaneously. Hence the problem has no solution because the constraints are inconsistent.

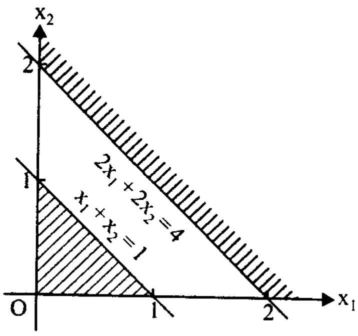


Fig. 3.12.

Example 35. ***(Constraints can be consistent and yet there may be no solution)***

Max. $z = x_1 + x_2$ *subject to* $x_1 - x_2 \geq 0, -3x_1 + x_2 \geq 3,$ *and* $x_1, x_2 \geq 0$.

Graphical Solution. Fig. 3.13 shows that there is no region of feasible solutions in this case. Hence there is no feasible solution. So the question of having optimal solution does not arise.

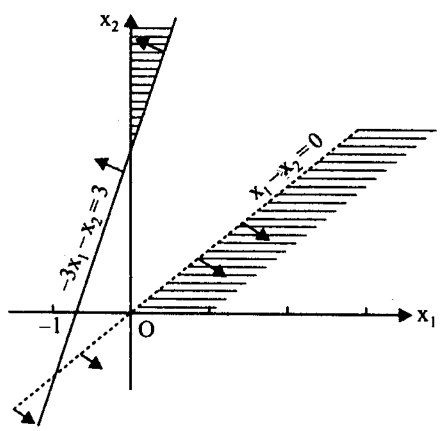


Fig. 3.13

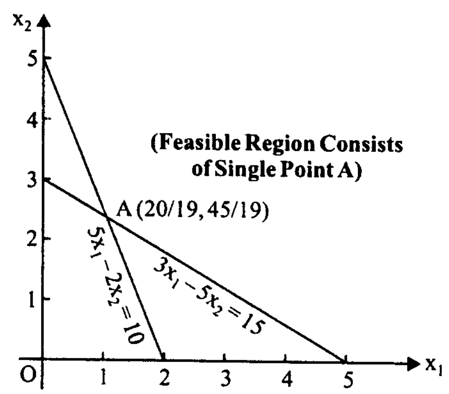


Fig. 3.14.

Example 36. ***(Problem in which constraints are equations rather than inequalities)***

Max. $z = 5x_1 + 3x_2$ *subject to* $3x_1 + 5x_2 = 15, 5x_1 + 2x_2 = 10, x_1 \geq 0, x_2 \geq 0$

Graphical Solution. Fig. 3.14 shows the graphical solution. Since there is only a single solution point A (20/19, 45/19), there is nothing to be maximized. Hence, a problem of this kind is of no importance. Such problems can arise only when the number of equations in the constratints is at least equal to the number of variables. If the solution is feasible, it is optimal. If it is not feasible, the problem has no solution.

Example 37. *A firm plans to purchase at least 200 quitals of scrap containing high quality metal X and low quality metal Y. It decides that the scrap to be purchased must contain at least 100 quintal of X-metal and not more than 35 quintals of Y-metal. The firm can purchase the scrap from two suppliers (A and B) in unlimited quantites. The percentage of X and Y metals in terms of weight in the scraps supplied by A and B is given below :*

Metals	Supplier A	Supplier B
X	25%	75%
Y	10%	20%

The price of A's scrap is Rs. 200 per quintal and that of B's is Rs. 400 per quintal. Formulate this problem as LP model and solve it to determine the quantities that the firm should buy from the two suppliers so as to minimize total purchase cost. **[Delhi (MBA) 98]**

Solution. The formulation of the given problem is :

Min. (total cost) $Z = 200x_1 + 400x_2$,

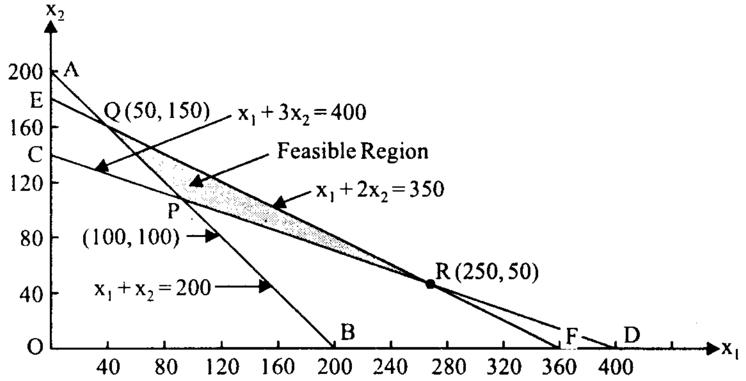


Fig. 3.15.

subject to the constraints :

$$x_1 + x_2 \geq 200, \frac{1}{4}x_1 + \frac{3}{4}x_2 \geq 100, \frac{1}{10}x_1 + \frac{1}{5}x_2 \leq 35, x_1 \geq 0, x_2 \geq 0.$$

where x_1, x_2 represent the number of quintals of scrap from two suppliers A and B respectively.

The feasible region is the shaded area PQR which is obtained by drawing the graph of the constraints :

$$x_1 + x_2 \geq 200, x_2 + 3x_2 \geq 400 \text{ and } x_1 + 2x_2 \leq 175$$

The coordinates of the corner points of the feasible region are :

$$P (100, 100), Q (50, 150), R (250, 50)$$

The Z has the min. value at the point P (100, 100). Thus the answer is $x_1 = 100, x_2 = 100$, min. Z = Rs. 60,000.

Example 38. *The standard weight of a special purpose brick is 5 kg and it contains two basic ingredients B_1 and B_2. B_1 costs Rs. 5 per kg and B_2 costs Rs. 8 per kg. Strength considerations state that the brick contains not more than 4 kg of B_1 and minimum of 2 kg of B_2. Since the demand for the product is likely to be related to the price of the brick, find out graphically minimum cost of the brick satisfying the above conditions.*

Solution. The formulation of the given problem is :

Min (total cost) $Z = 5x_1 + 8x_2$,

subject to the constraints :

$x_1 \le 4, x_2 \ge 2$ and $x_1 + x_2 = 5, x_1 \ge 0, x_2 \ge 0$, where (x_1, x_2) = the amount of ingredients B_1 (in kg) and B_2 (in kg.) respectively. The given constraints are plotted on the graph as shown in the figure. It may be observed that feasible region has two corner points $P(3, 2)$ and $Q(4, 2)$. The minimum value of Z is found at $P(3, 2)$, *i.e.* $x_1 = 3, x_2 = 2$. Hence the optimum product mix is to have 3 kg. of ingredient B_1 and 2 kg. of ingredient B_2 of a special case brick in order to achieve the minimum cost of Rs. 31.

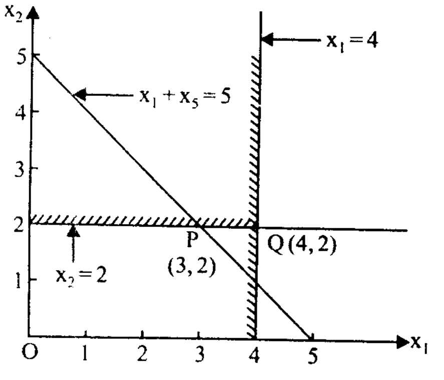


Fig. 3.16.

Application of LP on Management Accounts

Example 39. *A local travel agent is planning a charter trip to a major sea resort. The eight-day seven-night package includes the fare for round-trip travel, surface transportation, board and lodging and selected tour options. The charter trip is restricted to 200 persons and past experience indicates that there will not be any problem for getting 200 persons. The problem for the travel agent is to determine the number of Deluxe, Standard, and Economy tour packages to offer for this charter. These three plans each differ according to seating and service for the flight, quality of accomodation, meal plans and tour options. The following table summarizes the estimated prices for the three packages and the corresponding expenses for the travel agent. The travel agent has hired an air craft for the flat fee of Rs. 2,00,000 for the entire trip.*

Price and Costs for four packages per person

Tour plan	Price (Rs.)	Hotel costs (Rs.)	Meals & other expenses (Rs.)
Deluxe	10,000	3,000	4,750
Standard	7,000	2,200	2,500
Economy	6,500	1,900	2,200

In planning the trip, the following considerations must be taken into account :

(i) At least 10 per cent of the packages must be of the deluxe type.

(ii) At least 35 per cent but not more than 70 per cent must be of the standard type.

(iii) At least 30 per cent must be of the economy type.

(iv) The maximum number of deluxe packages available in any air craft is restricted to 60.

(v) The hotel desires that at least 120 of the tourists should be on the deluxe and standard packages together.

The travel agent wishes to determine the number of packages to offer in each type so as to maximize the total profit.

(a) Formulate the above as a linear programming problem.

(b) Restate the above linear programming problem in terms of two decision variables, taking advantage of the fact that 200 packages will be sold.

(c) Find the optimum solution using graphical methods for the restated linear programming problem and interpret your results. **[C.A. (May 91)]**

Solution. Let x_1, x_2, x_3 be the number of Deluxe, Standard & Economy tour packages restricted to 200 persons only to maximize the profits of the concern.

The contribution (per person) arising out of each type of tour package offered is as follows :

Package type offered	Price (Rs.)	Hotel Costs (Rs.)	Meals, etc. (Rs.)	Net profit (Rs.)
	(1)	(2)	(3)	(4) = (1) – [(2) + (3)]
Deluxe	10,000	3,000	4,750	2,250
Standard	7,000	2,200	2,500	2,300
Economy	6,500	1,900	2,200	2,400

Since the travel agent has to pay the flat fee of Rs. 2,00,000 for the chartered aircraft for the entire trip, the profit function will be :

$$\text{Max. } P = \text{Rs. } (2250x_1 + 2300x_2 + 2400x_3) - \text{Rs. } 2{,}00{,}000.$$

The constraints according to given conditions (i) to (v) are as follows :

$x_1 \geq 20$ from (i) $\quad x_3 \geq 60$ from (iii) $\quad x_1 + x_2 + x_3 = 200,$

$x_2 \geq 70$ from (ii) $\quad x_1 \leq 60$ from (iv)

$x_2 \leq 140$ from (ii) $\quad x_1 + x_2 \geq 120$ from (v) $\quad$ and $\quad x_1, x_2, x_3 \geq 0$

The compact form, above constraints can be reduced to the following forms :

$$20 \leq x_1 \leq 60,\ 70 \leq x_2 \leq 140, x_3 \geq 60, x_1 + x_2 \geq 120, x_1 + x_2 + x_3 = 200 \textit{ and } x_1, x_2, x_3 \geq 0, \text{ is}$$

(a) The linear programming formation is as given above.

(b) Since $x_1 + x_1 + x_3 = 200$, *i.e.* $x_3 = 200 - (x_1 + x_2)$, substitute the value of x_3 in the above relations to get the following reduced LPP :

Max. $P = -150x_1 - 100x_2 + 2{,}80{,}000$, subject to $20 \leq x_1 \leq 60$, $70 \leq x_2 \leq 140$, $120 \leq x_1 + x_2 \leq 140$ and $x_1, x_2 \geq 0$.

(c) **Graphical Solution.** Refer to the following figure for the restated LP. problem in (b).

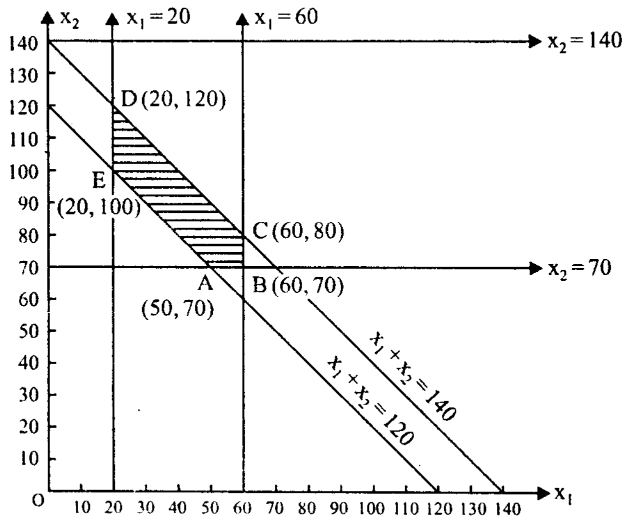


Fig. 3.17.

From above figure, we compute

Corner points	Coordinates of corner points	Values of objective function : $P = -150x_1 - 100x_2 + 2{,}80{,}000$
A	(50, 70)	P_A = Rs. 2,65,500
B	(60, 70)	P_B = Rs. 2,64,000
C	(60, 80)	P_C = Rs. 2,63,000
D	(20, 120)	P_D = Rs. 2,65,000
E	(20, 100)	P_E = Rs. 2,67,000

Thus maximum profit is attained at the corner point (20, 100).

Interpretation of Solution. Maximum profit of Rs. 2,67,000 is attained when $x_1 = 20, x_2 = 100$ and $x_3 = 200 - (x_1 + x_2) = 80$.

In other words, the travel agent should offer 20 *Delux*, 100 *Standard* and 80 *Economy* tour packages so as to get the maximum profit of Rs. 2,67,000.

Example 40. *Semicond is an electronics company manufacturing tape recorders and radios. Its per unit labour costs, raw material costs and selling prices are given in Table 1. An extract from its balance sheet on 31.3.1994 is shown in Table 2. Its current asset/current liability ratio (called the current ratio) is 2.*

Table 1 : *Cost Information*

For Products	Selling Price	Labour Cost	Raw Material Cost
Tape Recorder	Rs. 1,000	Rs. 500	Rs. 300
Radio	Rs. 900	Rs. 350	Rs. 400

Table 2 : *Extract from Balance Sheet as on 31.3.1994*

	Current Liabilities (Rs.)		Current Assets (Rs.)
	Cash		1,00,000
	* Accounts Receivable		30,000
	** Inventory		70,000
	Short-Term Bank Borrowing	1,00,000	

** Accounts receivable is amount due from customers.*

*** 100 units of raw material used for tape recorder and 100 units of raw material used for radio.*

Semicond must determine how many tape recorders and radios should be produced during April 94. Demand is large enough to ensure that all goods produced will be sold. All sales are on credit and payment for goods sold in April 94 will not be received until 31.5.94. During April 94, it will collect Rs. 20,000 in accounts receivable and it must payoff Rs. 10,000 of the outstanding short term bank borrowing and a monthly rent of Rs. 10,000. On 30.4.94, it will receive a shipment of raw material worth Rs. 20,000, which will be paid on May 31, 1994. The management has decided that the cash balance on April 30, 1994 must be at least 40,000. Also its banker requires that the current ratio as on April 30, 94 be at least 2. In order to maximize the contribution to profit for April 94 production it has to find the product mix for April 94. Assume that labour costs (wages) are paid in the month in which they are incurred. Formulate this as a linear programming problem and graphically solve it. **[C.A. (Nov. 94)]**

Solution. *Formutation.* Let x_1 and x_2 denote the number of units of tape recorders and radios respectively to be produced during April 1994.

Profit per unit of tape recorder = Selling price − (Labour cost + Raw material cost)
= Rs. 1000 − (Rs. 500 + Rs. 300) = Rs. 200

Similarly, profit per unit of radio = Rs. 900 − (Rs. 350 + Rs. 400) = Rs. 150

Company wishes to maximize its profit, therefore objective function is :

Maximize $\mathbf{P} = 200x_1 + 150x_2$, subject to the following constraints :

(1) As per data given in the balanced sheet, the inventory available in the stock can be used only to produce 100 units of tape recorder and 100 units of radio. Therefore, $x_1 \leq 100$ and $x_2 \leq 100$.

(2) The management has decided that the cash balance on April 30, 1994 must be at least Rs. 40,000.

Cash balance = Cash in hand on March 31, 94 + Accounts receivable collected in April 94
− Bank borrowing paidoff in April − Monthly rent paid
− Labour cost paid during April 94.
= Rs. 1,00,000 + Rs. 20,000 − Rs. 10,000 − Rs. 10,000 − $(500x_1 + 350x_2)$

The management wants cash balance on (April 30, 1994) ≥ Rs. 40,000, *i.e.* ...(i)

$$\text{Rs. } 1{,}00{,}000 - 500x_1 - 350x_2 \geq \text{Rs. } 40{,}000$$

or $$\text{Rs. } 60{,}000 \geq 500x_1 + 350x_2 \quad \text{or} \quad 500x_1 + 350x_2 \leq \text{Rs. } 60{,}000 \qquad \text{...(ii)}$$

(3) Bankers require that current ratio as on (April 30, 1994) ≥ 2,

Current ratio = current assets/current liabilities

Now we have to find the value of cash balance, accounts receivable, inventory and current liabilities as on April 30, 1994.

Cash balance = Rs. 1,00,000 − $500x_1 - 350x_2$. ...[from (i)]

Accounts receivables as on April 30, 1994 = Accounts receivable on March 31, 1994
+ Accounts receivable due from April sale − Accounts receivable collected during April
= Rs. 30,000 + $(1000x_1 + 900x_2)$ − Rs. 20,000 = Rs. 10,000 + $1000x_1 + 900x_2$

Inventory on April 30, 1994 = Inventory as on March 31, 1994 + Inventory received during April, 1994
– Inventory consumed during April, 1994

$= \text{Rs. } 70{,}000 + \text{Rs. } 20{,}000 - (300x_1 + 400x_2) = \text{Rs. } 90{,}000 - (300x_1 + 400x_2)$

Current assets as on April 30, 1994 = Cash balance + Accounts receivables
+ Inventory value on April 30, 1994

$= \text{Rs. } 1{,}00{,}000 - 500x_1 - 350x_2 + \text{Rs. } 10{,}000 + 1000x_1 + 900x_2 + \text{Rs. } 90{,}000 - 300x_1 - 400x_2$

$= \text{Rs. } 2{,}00{,}000 + 200x_1 + 150x_2$

Current liabilities as on April 30, 1994 = Value of bank borrowings as on March, 94
– Loan paid during April, 1994
+ Amount due on inventory received during April 1994

$= \text{Rs. } 1{,}00{,}000 - \text{Rs. } 10{,}000 + \text{Rs. } 20{,}000 = \text{Rs. } 1{,}10{,}000.$

But bank requires that current ratio as on April 30, 1994 be at least 2.

That is, current assets/current liabilities ≥ 2 or $\left(\dfrac{\text{Rs. } 2{,}00{,}000 + 200x_1 + 150x_2}{\text{Rs. } 1{,}10{,}000}\right) \geq 2$

or $\text{Rs. } 2{,}00{,}000 + 200x_1 + 150x_2 \geq \text{Rs. } 2{,}20{,}000$ or $200x_1 + 150x_2 \geq \text{Rs. } 20{,}000.$ ...(iii)

Thus the linear programming model for the Semicond is as follows :

Maximize $P = 200x_1 + 150x_2$, subject to the constraints :

$$x_1 \leq 100,\ x_2 \leq 100,\ 500x_1 + 350x_2 \leq 60000,\ 200x_1 + 150x_2 \geq 20000$$

Graphical Solution. The feasible region enclosed by the constraints is given by points A, B, C, D with coordinates :

$$A(25, 100),\ B(50, 100),\ C(100, \tfrac{200}{7}),\ D(100,0)$$

The profit at these coordinates is found below :

$A(25, 100)$: Rs. 200 × 25 + Rs. 150 × 100 = Rs. 20,000

$B(50, 100)$: Rs. 200 × 50 + Rs. 150 × 100 = Rs. 25,000

$C(100, 200/7)$: Rs. 200 × 100 + Rs. 150 × 200/7 = Rs. 24285·7

$D(100, 0)$: Rs. 200 × 100 + Rs. 150 × 0 = Rs. 20,000.

Since maximum profit is attained at the point $B(50, 100)$, Semicond can maximize its profit by producing 50 tap recorders and 100 radios during April, 1994 and the total profit contribution will be Rs. 25,000.

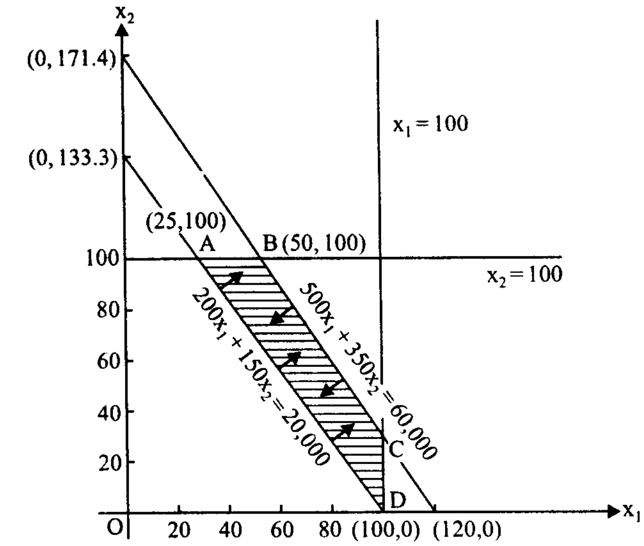


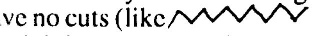
Fig. 3.18.

Q. Explain (i) No feasible solution, (ii) Unbounded solution. Give one example in each case.

3.3-4. Important Geometric Properties of LP Problems

Geometric properties of LP problems, which are observed while solving them graphically, are summarized below :

1. The region of feasible solutions has an important property which is called the *convexity property* in geometry, provided the feasible solution of the problem exists.

Convexity means that region of feasible solutions has *no holes* in them, that is, they are solids, and they have no cuts (like ) on the boundary. This fact can be expressed more precisely by saying that the line joining any two points in the region also lies in the region.

2. The boundaries of the regions are lines or planes.

3. There are corners or exterme points on the boundary, and there are edges joining various corners.

4. The objective function can be represented by a line or a plane for any fixed value of z.

5. At least one corner of the region of feasible solutions will be an optimal solution whenever the maximum or minimum value of z is finite.

6. If the optimal solution is not unique, there are points other than corners that are optimal but in any case at least one corner is optimal.

7. The different situation is found when the objective function can be made arbitrarily large. Of course, no corner is optimal in that case.

EXAMINATION PROBLEMS

1. Solve the following LP problems by graphical method :

(a) Min. $z = 5x_1 - 2x_2$; s.t. $2x_1 + 3x_2 \geq 1$, $x_1, x_2 \geq 0$.
[Hint. Vertices of the feasible region are : $(\frac{1}{2}, 0)$, $(0, \frac{1}{3})$]
[Ans. $x_1 = 0$, $x_2 = 1/3$, min. $z = -2/3$].

(b) Max. $z = 5x_1 + 3x_2$; s.t. $3x_1 + 5x_2 \leq 15$, $5x_1 + 2x_2 \leq 10$; $x_1, x_2 \geq 0$
[Hint. Vertices of the feasible region are : (0, 0), (2, 0), (20/19, 45/19) and (0, 3).]
[Ans. $x_1 = 20/19$, $x_2 = 45/19$, max. $z = 235/19$]

(c) Max. $z = 2x_1 + 3x_2$; s.t. $x_1 + x_2 \leq 1$, $3x_1 + x_2 \leq 4$: $x_1, x_2 \geq 0$.
[Hint. Vertices of the feasible region are : (0, 0), (1, 0), (0, 1).]
[Ans. $x_1 = 0$, $x_2 = 1$, max. $z = 3$]

(d) Max. $z = 5x_1 + 7x_2$; s.t. $x_1 + x_2 \leq 4$, $3x_1 + 8x_2 \leq 24$, $10x_1 + 7x_2 \leq 35$, $x_1, x_2 \geq 0$. **[Meerut 90]**
[Hint. Vertices of the feasible region are : (0, 0), (7/2, 0), (7/3, 5/3), (8/5, 12/5) and (0, 3).]
[Ans. $x_1 = 8/5$, $x_2 = 12/5$, max. $z = 124/5$]

(e) Min. $z = -x_1 + 2x_2$; s.t. $-x_1 + 3x_2 \leq 10$, $x_1 + x_2 \leq 6$, $x_1 - x_2 \leq 2$, $x_1, x_2 \geq 0$.
[Hint. Vertices of the feasible region are : (0, 0) (2, 0), (4, 2), (2, 4) and (0, 10/3).]
[Ans. $x_1 = 2$, $x_2 = 0$, min. $z = -2$]

(f) Min. $z = 20x_1 + 10x_2$; s.t. $x_1 + 2x_2 \leq 40$, $3x_1 + x_2 \geq 30$, $4x_1 + 3x_2 \geq 60$, and $x_1 \geq 0$, $x_2 \geq 0$.
[Hint. Vertices of the feasible region are : (15, 0) (40, 0), (4, 18) and (6, 12).]
[Ans. $x_1 = 6$, $x_2 = 12$, min. $z = 240$].

(g) Max. $z = 3x_1 + 4x_2$; s.t. $x_1 - x_2 \leq -1$, $-x_1 + x_2 \leq 0$; $x_1, x_2 \geq 0$. **[Bhubneshwar (IT) 2004]**
[Ans. The problem has no solution]

(h) Max. $z = 3x + 2y$; $-2x + 3y \leq 9$, $x - 5y \geq -20$; $x, y \geq 0$.
[Ans. The problem has an unbounded solution].

(i) Max. $z = x_1 + 3x_2$; $3x_1 + 6x_2 \leq 8$, $5x_1 + 2x_2 \leq 10$; $x_1, x_2 \geq 0$.
[Hint. The vertices of the feasible region are : (0, 0), (2, 0), (11/6, 5/12), (0, 4/3)
[Ans. $x_1 = 0$, $x_2 = 4/3$, max. $z = 4$]

(j) Max. $z = 7x_1 + 3x_2$; s.t. $x_1 + 2x_2 \geq 3$, $x_1 + x_2 \leq 4$, $0 \leq x_1 \leq 5/2$, $0 \leq x_2 \leq 3/2$. **[IPM (PGDBM) 2000]**
[Hint. The vertices of the feasible region are : (0, 0), (5/2, 1/4), (5/2, 3/2), and (0, 3/2).]
[Ans. $x_1 = 5/2$, $x_2 = 3/2$, max. $z = 22$].

(k) Max. $z = 3x + 4y$; s.t. $4x + 8y \leq 32$, $9x + 2y \geq 14$, $3x/2 + 5y \geq 15$, where $x, y \geq 0$.
[Hint. Vertices are : (3/4, 29/8), (2/3, 19/7), (0, 4), (0, 3).]
[Ans. $x = 3/4$, $y = 29/8$, max. $z = 17{\cdot}2$]

(i) Max. $z = 2x_1 + x_2$, s.t. $x_1 + 2x_2 \le 10$, $x_1 + x_2 \le 6$, $x_1 - x_2 \le 2$, $x_1 - 2x_2 \le 1$, $x_1, x_2 \ge 0$. **[IPM (PGDBM) 2000]**
[**Hint.** The vertices of the feasible region are : (0, 0), (1, 0), (3, 1), (4, 2), (2, 4) and (0, 5)]
[**Ans.** $x_1 = 4$, $x_2 = 2$, max. $z = 10$]

2. Does the following LPP has a feasible solution ? Max. $z = x_1 + x_2$, subject to $x_1 - x_2 \ge 0$, $3x_1 - x_2 \le -3$. Show with the help of a graph.
[**Ans.** No feasible solution.]

3. Solve the following LPP's graphically :

(a) Maximize $z = 45x_1 + 80x_2$,
subject to the constraints :
$5x_1 + 20x_2 \le 400$
$10x_1 + 15x_2 \le 450$
$x_1, x_2 \ge 0$
[**Ans.** $x_1 = 24$, $x_2 = 14$, max. $z = 2200$]

(b) Minimize $z = 7y_1 + 8y_2$,
subject to the constraints :
$3y_1 + y_2 = 8$
$y_1 + 3y_2 \ge 11$
$y_1, y_2 \ge 0$
[**Ans.** $y_1 = 13/8$, $y_2 = 25/8$, min. $z = 291/8$]

(c) Maximize $z = 30x + 40y$.
subject to the constraints :
$50x + 36y \le 1,00,000$,
$25x + 36y \le 91,000$,
$x, y \ge 0$
[**Ans.** $x = 360$, $y = 2277{\cdot}77$, max. $z = 1,01,911$]

(d) Minimize $z = -6x_1 - 4x_2$,
subject to the constraints :
$2x_1 + 3x_2 \ge 30$,
$3x_1 + 2x_2 \le 24$,
$x_1 + x_2 \ge 3$
$x_1, x_2 \ge 0$
[**Ans.** (i) $x_1 = 8$, $x_2 = 0$,
(ii) $x_1 = 12/5$, $x_2 = 42/5$, min. $z = -48$]

(e) Maximize $z = 3x_1 + 2x_2$,
subject to the constraints :
$2x_1 - x_2 \ge -2$
$x_1 + 2x_2 \ge 8$
$x_1, x_2 \ge 0$
[**Ans.** Unbounded solution]

(f) Maximize $z = 5x_1 + 7x_2$,
subject to the constraints :
$x_1 + x_2 \le 4$, $3x_1 + 8x_2 \le 24$
$10x_1 + 7x_2 \le 35$
$x_1, x_2 \ge 0$
[**Ans.** $x_1 = 8/5$, $x_2 = 12/5$, max. $z = 124/5$]

(g) Maximize $z = 120x_1 + 100x_2$,
subject to the constraints :
$10x_1 + 5x_2 \le 80$
$6x_1 + 6x_2 \le 66$
$4x_1 + 8x_2 \ge 24$
$5x_1 + 6x_2 \le 90$
$x_1, x_2 \ge 0$.
[**Ans.** $x_1 = 500$, $x_2 = 600$, max. $z = 1200$]

(h) Maximize $z = 2x_1 + 3x_2$,
subject to the constraints :
$x_1 + x_2 \ge 1$, $5x_1 - x_2 \ge 0$
$x_1 + x_2 \le 6$
$x_1 - 5x_2 \le 0$
$x_2 - x_1 \ge -1$
$x_2 \le 3$; $x_1, x_2 \ge 0$
[**Ans.** $x_1 = 4$, $x_2 = 3$, max. $z = 17$]
[JNTU (MCA III) 2004, (B. Tech.) 2003]

4. Find the maximum and minimum value of $z = 5x_1 + 3x_2$, subject to the constraints : $x_1 + x_2 \le 6$, $2x_1 + 3x_2 \ge 3$, $0 \le x_1 \ge 3$, $0 \le x_2 \ge 3$.
[**Ans.** $x_1 = 3$, $x_2 = 3$, min. $z = 24$].

5. Solve the LPP given below by graphical method and shade the region representing the feasible solution :
$x_1 - x_2 \ge 0$, $x_1 - 5x_2 \ge -5$, and $x_1, x_2 \ge 0$, minimum $z = 2x_1 - 10x_2$.
[**Ans.** $x_1 = 5/4$, $x_2 = 5/4$, min. $z = -10$]

6. Using graphical method find non-negative values of x_1 and x_2, which :

(a) Maximize $z = x_1 + 2x_2$, subject to the constraints :
$x_1 + x_2 \le 6$ $x_1 + x_2 \le 2$, $x_1 + 3x_2 \ge 6$, $-x_1 + 3x_2 \le 10$.
[**Ans.** $x_1 = 0$, $x_2 = 2$, max. $z = 4$]

(b) Minimize : $z = 600x_1 + 400x_2$,
subject to the constraints : $1500x_1 + 1500x_2 \ge 20{,}000$, $3000x_1 + 1000x_2 \ge 40{,}000$, $2000x_1 + 5000x_2 \ge 44{,}000$.
[**Ans.** $x_1 = 12$, $x_2 = 4$, min. $z = 8800$.]

7. Solve graphically the following LP problem :
Min. $z = 3x_1 + 5x_2$, *s.t.* $-3x_1 + 4x_2 \le 12$, $2x_1 - x_2 \ge -2$, $2x_1 + 3x_2 \ge 12$, $x_1 \le 4$, $x_2 \ge 2$, $x_1, x_2 \ge 0$ **[Meerut 91]**
[**Ans.** $x_1 = 3$, $x_2 = 2$, min. $z = 19$]

8. Mark the feasible region represented by the constraint equations :
$x_1 + x_2 \le 1$, $3x_1 + x_2 \ge 3$, $x_1 \ge 0$, $x_2 \ge 0$ of a linear optimizing function $z = x_1 + x_2$.

9. Find optimum solution of the following problem by not using artificial variables :
Min. $z = 10x_1 + 10x_2$, s.t. $x_1 + x_2 \ge 10$, $3x_1 + 2x_2 \ge 24$, $x_1 \ge 0$, $x_2 \ge 0$. **[Delhi (MCI) 2000]**

10. Find the feasible zone for the constraints $x_1 + x_2 \geq 1$, $x_1 + 2x_2 \geq 6$, $x_1 - x_2 \leq 3$, $x_1 \geq 0$ and state the redundant constraints. If $z = 3x_1 + x_2$, draw two iso-z lines and show the direction of improvement if z is to be minimized. Find z_{min}. Also state what can be the maximum value of z. **[V.T.U. (BE Mech.) 2002]**

11. Solve the following LPP :

    Maximize $z = 10x + 15y$, subject to $x + y \leq 30$, $y \geq 3$, $x - y \geq 0$, $y \leq 12$, $x \leq 20$ and $x, y + 0$. **[JNTU (B. Tech.) 2003]**

12. A firm makes two types of furniture chairs and tables. The contribution for each product as calculated by the accounting department is Rs. 20/- per chair and Rs. 30/- per table. Both products are processed on three machines M_1, M_2, M_3. The time required in hours by each product and total time available in hours per week on each machine are as follows :

Machine	Chairs	Table	Available Time
M_1	3	3	36
M_2	5	2	50
M_3	2	6	60

How should the manufacturer schedule his production in order to maximize contribution? Solve graphically.
[VTU (BE Mech.) 2002]

3.4. GENERAL FORMULATION OF LINEAR PROGRAMMING PROBLEM

The general formulation of the linear programming problem can be stated as follows :

In order to find the values of n decision variables $x_1, x_2, ..., x_n$ to maximize or minimize the objective function

$$z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n \quad ...(3.7)$$

and also satisfy m-constraints :

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n \ (\leq = \text{or} \geq)\ b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n \ (\leq = \text{or} \geq)\ b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n \ (\leq = \text{or} \geq)\ b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n \ (\leq = \text{or} \geq)\ b_m, \end{array}\right\} \quad ...(3.8)$$

where constraints may be in the from of any inequality ($\leq$ or $\geq$) or even in the form of an equation (=), and finally satisfy the non-negativity restrictions

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_j \geq 0, \ldots, x_n \geq 0. \quad ...(3.9)$$

However, by convention, the values of right side parameters b_i ($i = 1, 2, 3 \ldots, m$) are restricted to non-negative values only. It is important to note that any negative b_i can be changed to a positive value on multiplying both sides of the constraint by -1. This will not only change the sign of all left side coefficients and right side parameters but will also change the direction of the inequality sign.

Q. 1. What do you mean by a L.P.P.? What are its limitations ?

2. Define a general linear programming problem. **[Meerut (L.P.) 90]**

3. What is linear programming problem (LPP) ? How can formulate a given problem into LPP ?
[IGNOU 2001, 2000, 98, 97, 96]

3.5 SLACK AND SURPLUS VARIABLES

1. Slack Variables. If a constraint has $\leq$ sign, then in order to make it an equality, we have to add something positive to the left hand side. **[JNTU (B. Tech.) 2003; Jaunpur (B.Sc.) 96; Meerut 90]**

The non-negative variable which is added to the left hand side of the constraint to convert it into equation is called the slack variable.

For example, consider the constraints :

$$x_1 + x_2 \leq 2,\ 2x_1 + 4x_2 \leq 5,\ x_1, x_2 \geq 0 \quad ...(i)$$

We add the slack variables $x_3 \geq 0$, $x_4 \geq 0$ on the left hand sides of above inequalities respectively to obtain

$$\begin{array}{l} x_1 + x_2 + x_3 \quad\quad = 2 \\ 2x_1 + 4x_2 \quad + x_4 = 5 \\ x_1, x_2, x_3, x_4 \geq 0. \end{array}$$

2. Surplus Variables. If a constraint has ≥ sign, then in order to make it an equality, we have to subtract something non-negative from its left hand side.

Thus the positive variable which is subtracted from the left hand side of the constraint to convert it into equation is called the surplus variable.

For example, consider the constraints :

$$x_1 + x_2 \geq 2,\ 2x_1 + 4x_2 \geq 5, \text{ and } x_1, x_2 \geq 0. \qquad \text{...(ii)}$$

We subtract the surplus variables $x_3 \geq 0$, $x_4 \geq 0$ from the left hand sides of above inequalities respectively to obtain

$$\begin{aligned} x_1 + x_2 - x_3 &= 2 \\ 2x_1 + 4x_2 - x_4 &= 5 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

3.6. STANDARD FORM OF LINEAR PROGRAMMING PROBLEM

The standard form of the linear programming problem is used to develop the procedure for solving general linear programming problem. The characteristics of the standard form are explained in the following steps :

Step 1. ***All the constraints should be converted to equations except for the non-negativity restrictions which remain as inequalities*** (≥ 0). Constraints of the inequality type can be changed to equations by augmenting (adding or subtracting) the left side of each such constraint by non-negative variables. These new variables are called ***slack variables*** and are added if the constraints are (≤) or subtracted if the constraints are (≥). Since in the case of ≥ constraint, the subtraced variable represents the surplus of the left side over the right side, it is common to refer to it as surplus variable. For convenience, however, the name 'slack' variable will also be used to represent this type of variable. In this respect, a surplus is regarded as a negative slack.

For example, consider the constraints : $3x_1 - 4x_2 \geq 7,\ x_1 + 2x_2 \leq 3$.

These constraints can be changed to equations by introducing slack variables x_3 and x_4 respectively. Thus, we get

$$3x_1 - 4x_2 - x_3 = 7,\ x_1 + 2x_2 + x_4 = 3, \text{ and } x_3 \geq 0, x_4 \geq 0.$$

Step 2. ***The right side element of each constraint should be made non-negative (if not).*** The right side can always be made positive on multiplying both sides of the resulting equation by (– 1) whenever it is necessary.

For example, consider the constraint as $3x_1 - 4x_2 \geq -4$

which can be written in the form of the equation $3x_1 - 4x_2 - x_3 = -4$

by introducing the surplus variable $x_3 \geq 0$.

Again, multiplying both sides by (– 1), we get $-3x_1 + 4x_2 + x_3 = 4$ which is the constraint equation in standard form.

Step 3. ***All variables must have non-negative values.***

A variable which is *unrestricted in sign* (that is, positive, negative or zero) is equivalent to the difference between two non-negative variables. Thus, if x is unconstrained in sign, it can be replaced by $(x' - x'')$, where x' and x'' are both non-negative, that is, $x' \geq 0$ and $x'' \geq 0$.

Step 4. ***The objective function should be of maximization form.***

The minimization of a function $f(x)$ is equivalent to the maximization of the negative expression of this function, $f(x)$, that is,

$$\text{Min. } f(x) = -\text{Max } [-f(x)]$$

For example, the linear objective function

$$\text{Min. } z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \qquad \text{...(3.10)}$$

is equivalent to Max $(-z)$, *i.e.* Max $z' = -c_1x_1 - c_2x_2 - \ldots - c_nx_n$ with $z = -z'$.

Consequently, in any L.P problem, the objective function can be put in the maximization form.

Standard Form of General LPP with '≤' Constraints :

Now, applying above steps systematically to general form of L.P. problem with all (≤) constraints, the following standard form is obtained. Of course, no difficulty will arise to convert the general LPP with mixed constraints (≤ = ≥).

$$\text{Max. } z = c_1x_1 + c_2x_2 + \ldots + c_nx_n + 0x_{n+1} + \ldots + 0x_{n+m} \qquad \ldots(3.11)$$

subject to

$$\left.\begin{array}{llll} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + x_{n+1} & & & = b_1 \\ a_{21}x_2 + a_{22}x_2 + \ldots + a_{2n}x_n & + x_{n+2} & & = b_2 \\ \vdots \quad \vdots \quad \vdots & & \ddots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & & + x_{n+m} & = b_m \end{array}\right\} \qquad \ldots(3.12)$$

where $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0, x_{n+1} \geq 0, \ldots, x_{n+m} \geq 0.$ ...(3.13)

Note.

1. It should be remembered that the coefficient of slack variables $x_{n+1}, x_{n+2}, \ldots x_{n+m}$ in the objective function are assumed to be zero, so that the conversion of constraints to a system of simultaneous linear equations does not change the function to be optimized.
2. Since in the case of ($\geq$ 0) constraints, the subtracted variable represents the surplus variable. However, the name slack variable may also represent this type. In this respect, a surplus is regarded as a negative slack.

Q. Define slack and surplus variables as involved in the L.P.P, How are these variables useful in solving a L.P.P.?

[AIMS (Banglore) MBA 2002]

Example 33. *Express the following L.P. problem in standard form.*

Min. $z = x_1 - 2x_2 + x_3$, *subject to* :

$2x_1 + 3x_2 + 4x_3 \geq -4$, $3x_1 + 5x_2 + 2x_3 \geq 7$, $x_1 \geq 0$, $x_2 \geq 0$ *and* x_3 *is unrestricted in sign.*

Solution. Proceeding according to above rules, the standard LP form becomes :

Max $(z') = -x_1 + 2x_2 - (x_3' - x_3'')$, where $z' = -z$, subject to

$$-2x_1 - 3x_2 - 4(x_3' - x_3'') + x_4 = 4$$

$$3x_1 + 5x_2 + 2(x_3' - x_3'') - x_5 = 7$$

$$x_1 \geq 0, x_2 \geq 0, x_3' \geq 0, x_3'' \geq 0, x_4 \geq 0, x_5 \geq 0.$$

Of course, the number of variables will now increase to *six*.

3.7. MATRIX FORM OF LP PROBLEM

The linear programming problem in standard form [(3.11), (3.12), (3.13)] can be expressed in matrix form as follows :

Maximize $z = \mathbf{CX}^T$ *(objective function)*

subject to $\mathbf{AX} = \mathbf{b}, \mathbf{b} \geq \mathbf{0}$ *(constraint equation)*

$\mathbf{X} \geq \mathbf{0}.$ *(non-negativity restriction)*

where $\mathbf{X} = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$,

$\mathbf{C} = (c_1, c_2, \ldots, c_n, 0, 0, \ldots, 0)$, and $\mathbf{b} = (b_1, b_2, \ldots, b_m)$.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & 1 & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & 0 & 0 & \ldots & 1 \end{bmatrix}$$

Similar treatment can be adopted in the case of mixed constraints ($\leq$, = , $\geq$). Following example will make this point clear.

The vector $\mathbf{X}$ is assumed to include all decision variables, (i.e. original, slack and surplus). For convenience, $\mathbf{X}$ is used to represent all types of variables. The vector $\mathbf{C}$ gives the corresponding coefficients in the objective function. For example, if the variable is slack, its corresponding coefficient will be zero.

Example 42. *Express the following LP problem in the matrix form.*

Max. $z = 2x_1 + 3x_2 + 4x_3$, *subject to*

$x_1 + x_2 + x_3 \geq 5$, $x_1 + 2x_2 = 7$, $5x_1 - 2x_2 + 3x_3 \leq 9$, and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$.

Solution. This problem can be written in standard form as

Max. $z = 2x_1 + 3x_2 + 4x_3 + 0x_4 + 0x_5$ or Max. $z = (2, 3, 4, 0, 0)\,(x_1 x_2 x_3 x_4 x_5)^T$

subject to $x_1 + x_2 + x_3 - x_4 = 5$, $x_1 + 2x_2 = 7$, $5x_1 - 2x_2 + 3x_3 + x_5 = 9$

or $$\begin{bmatrix} 1 & 1 & 1 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 5 & -2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

Therefore, $\mathbf{X} = (x_1\ x_2\ x_3\ x_4\ x_5)^T$, $\mathbf{c} = (2\ 3\ 4\ 0\ 0)$, $\mathbf{b} = (5\ 7\ 9)^T$, and

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 5 & -2 & 3 & 0 & 1 \end{bmatrix}.$$

3.8 SOME IMPORTANT DEFINITIONS

Following are defined a few important terms for standard LPP (3.1, 3.2, 3.3) which are necessary to understand further discussion.

1. **Solution to LPP.** Any set $\mathbf{X} = \{x_1, x_2, \ldots, x_{n+m}\}$ of variables is called a ***solution*** to LP problem, if it satisfies the set of constraints (3.12) only.
2. **Feasible Solution (FS).** Any set $\mathbf{X} = \{x_1, x_2, \ldots, x_{n+m}\}$ of variables is called a ***feasible solution*** (or programme) of L.P. problem, if it satisfies the set of constraints (3.12) and non-negativity restrictions (3.13) also.
3. **Basic Solution (BS).** A ***basic solution*** to the set of constraints (3.12) is a *solution* obtained by setting any n variables (among $m + n$ variables) equal to zero and solving for remaining m variables, provided the determinant of the coefficients of these m variables is non-zero. Such m variables (of course, some of them may be zero) are called ***basic variables*** and remaining n zero-valued variables are called ***non-basic variables.*** **[Bhubnaeshwar (IT) 2004; JNTU (B. Tech) 98]**

   The number of basic solutions thus obtained will be at the most ${}^{m+n}C_m = \frac{(m+n)\,!}{n\,!\,m\,!}$, which is the number of combinations of $n + m$ things taken m at a time.
4. **Basic Feasible Solution (BFS).** A *basic feasible solution* is a *basic solution* which also satisfies the non-negativity restrictions (3.13), that is, all basic variables are non-negative. **[JNTU. MCA (III) 2004]**

   Basic feasible solutions are of two types :

   (a) ***Non-degenerate BFS.*** A non-degenerate basic feasible solution is the basic feasible solution which has exactly m positive x_i $(i = 1, 2, \ldots, m)$. In other words, all m basic variables are positive, and the remaining n variables will be all zero.

   (b) ***Degenerate BFS.*** A basic feasible solution is called *degenerate,* if one or more basic variables are *zero-valued.* **[JNTU. MCA (III) 2004]**
5. **Optimum Basic Feasible Solution.** A basic feasible solution is said to be ***optimum,*** if it also optimizes (maximizes or minimizes) the objective function (3.11). **[JNTU. MCA(III) 2004; Meerut (L.P) 90]**
6. **Unbounded Solution.** If the value of the objective function z can be increased or decreased indefinitely, such solutions are called ***unbounded solutions.*** **[Meerut 90]**

Note. Unless otherwise stated, solution means a feasible solution. However, an optimum solution to a linear programming problem imply that z has a finite maximum or finite minimum.

Q. 1. For the system AX = **b** of m linear equations in n unknowns $(m < n)$ with rank$(A) = m$, define a basic solution. **[Meerut (IPM) 91]**

2. Explain the term optimal solution to a LPP. **[AIMS (Bangalore) MBA 2002]**

3. Define :
(i) Feasible Solution
(ii) Basic Solution **[JNTU MCA (III) 2004; (B. Tech.) 98]**
(iii) Basic Feasible Solution **[VTU (BE Mech.) 2002; Kanpur 96]**
(iv) Non-dgenerate BFS **[JNTU. MCA (III) 2004]**
(v) Degenerate BFS
(vi) Optimum Basic Feasible Solution **[JNTU. MCA (III) 2004; Kanpur 96; Meerut (Stat.) 95 ; (Math.) 90]**
(vii) Unbounded Solution. **[JNTU (B. Tech) 98]**

Example 43. *Find all basic solutions for the system of simultaneous equations :*

$$2x_1 + 3x_2 + 4x_3 = 5 \quad \text{and} \quad 3x_1 + 4x_2 + 5x_3 = 6.$$

Solution. First decide the maximum number of basic solutions. The maximum possible number of basic solutions will be $^3C_2 = \frac{3\,!}{2\,!\,(3-2)\,!} = 3$.

Now, put $x_1 = 0$ and solve for x_2 and x_3. The values of x_2, x_3 thus obtained are : $x_2 = -1$, $x_3 = 2$.

Again, put $x_2 = 0$ and solve for x_1 and x_3. The values of x_1, x_3 are : $x_1 = -2$, $x_3 = \frac{3}{2}$.

Finally, put $x_3 = 0$ and solve for x_1 and x_2. The values of x_1, x_2 are : $x_1 = -2$. $x_2 = 3$.

This verifies that only three basic solutions exist which are *non-degenerate* and *infeasible* also.

EXAMINATION PROBLEMS

1. Determine all basic feasible solutions of the system of equations $2x_1 + x_2 + 4x_3 = 11$, $3x_1 + x_2 + 5x_3 = 14$.
   [Ans. (i) $x_1 = 3$, $x_2 = 5$, $x_3 = 0$; (ii) $x_1 = 1/2$, $x_2 = 0$, $x_3 = 5/2$]
2. For the following system of linear equations, determine all the extreme points and their corresponding basic solutions:
   $$3x_1 + x_2 + 5x_3 + x_4 = 12,\ 2x_1 + 4x_2 + x_3 + 2x_5 = 8,\ \text{where } x_1, x_2, x_3, x_4, x_5 \geq 0.$$
3. Find all the basic solutions of the equations : $2x_1 + 6x_2 + 2x_3 + x_4 = 3$, $6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$
   [JNTU (MCA III) 2004; Kanpur B.Sc. (II) 2003; Meerut (L.P.) 90]
   [Ans. (0, 0, 2, – 1), (8/3, 0, 0, – 7/3), (0, 1/2, 0, 0), (0, 1/2, 0, 0), (0, 1/2, 0, 0) and (– 2, 0, 7/2, 0)]
4. Show that the feasible solution $x_1 = 1$, $x_2 = 0$, $x_3 = 1$ and $z = 6$ to the system of equations : $x_1 + x_2 + x_3 = 2$, $x_1 - x_2 + x_3 = 2$, $x_j \geq 0$ $(j = 1, 2, 3)$ which minimize $z = 2x_1 + 3x_2 + 4x_3$, is not basic. **[Kanpur B.Sc. 95]**
5. Find all basic feasible solutions for the problem Max. $z = x_1 + 2x_2$ such that $x_1 + x_2 \leq 10$, $2x_1 - x_2 \leq 40$ and $x_1, x_2 \geq 0$.
   [VTU (BE Mech.) 2002]

3.5. ASSUMPTIONS IN LINEAR PROGRAMMING PROBLEM

As examined from above examples, following are the assumptions in linear programming problem that limit its applicability.

(a) Proportionality. A primary requirement of linear programming problem is that the objective function and every constraint function must be ***linear.*** Roughly speaking, it simply means that if 1 kg of a product costs Rs. 2, then 10 kg will cost Rs. 20. If a steel mill can produce 200 tons in 1 hour, it can produce 1000 tons in 5 hours.

Intuitively, linearity implies that the product of variables such as $x_1 x_2$, powers of variables such as x_3^2, and combination of variables such as $a_1 x_1 + a_2 \log x_2$, are not allowed.

(b) Additivity. As discussed in ***Example 1*** (page 54), additivity means if it takes t_1 hours on machine G to make product A and t_2 hours to make product B, then the time on machine G devoted to produce A and B both is $t_1 + t_2$, provided the time required to change the machine from product A to B is negligible.

The additivity may not hold, in general. If we mix several liquids of different chemical composition, then the total volume of the mixture may not be the sum of the volume of individual liquids.

(c) Multiplicativity. *It requires :*

(i) if it takes one hour to make a single item on a given machine, it will take 10 hours to make 10 such items; and

(ii) the total profit from selling a given number of units is the unit profit times the number of units sold.

(d) Divisibility. It means that the fractional levels of variables must be permissible besides integral values.

(e) Deterministic. All the parameters in the linear programming models are assumed to be known exactly. While in actual practice, production may depend upon chance also. Such type of problems, where some of the coefficients are not known, are discussed in the extension of sensitivity analysis known as parametric programming.

Significance of Assumptions :

A practical problem which completely satisfies all the above assumptions for linear programming is very rare indeed. Therefore, the user should be fully aware of the assumptions and appoximations involved and should satisfy himself that they are justified before proceeding to apply linear programming approach.

Q. 1. State clearly the basic assumptions that are made in LPP.
2. What are the major assumptions in Linear Programming ? [JNTU (Mech. & Prod.) 2004]

3.10. LIMITATIONS OF LINEAR PROGRAMMING

Inspite of wide area of applications, some limitations are associated with linear programming techniques. These are stated below :

1. In some problems objective functions and constraints are not linear. Generally, in real life situations concerning business and industrial problems constraints are not linearly treated to variables.

2. There is no guarantee of getting integer valued solutions, for example, in finding out how many men and machines would be required to perform a particular job, rounding off the solution to the nearest integer will not give an optimal solution. Integer programming deals with such problems.

3. Linear programming model does not take into consideration the effect of time and uncertainty. Thus the model should be defined in such a way that any change due to internal as well as external factors can be incorporated.

4. Sometimes large-scale problems cannot be solved with linear programming techniques even when the computer facility is available. Such difficulty may be removed by decomposing the main problem into several small problems and then solving them separately.

5. Parameters appearing in the model are assumed to be constant. But, in real life situations they are neither constant nor deterministic.

6. Linear programming deals with only single objective, whereas in real life situations problems come across with multiobjectives. *Goal programming* and ***multi-objective programming*** deal with such problems.

Q. What are the limitations of linear programming technique ?

3.11. APPLICATIONS OF LINEAR PROGRAMMING

In this section, we discuss some important applications of linear programming in our life.

1. Personnel Assignment Problem. Suppose we are given m persons, n-jobs, and the expected productivity c_{ij} of ith person on the jth job. We want to find an assignment of persons $x_{ij} \geq 0$ for all i and j, to n jobs so that the average productivity of person assigned is maximum, subject to the conditions :

$$\sum_{j=1}^{n} x_{ij} \leq a_i \text{ and } \sum_{i=1}^{m} x_{ij} \leq b_j,$$

where a_i is the number of persons in personnel category i and b_j is the number of jobs in personnel category j. For details, refer the chaper of *Assignment Problems.*

2. Transportation Problem. We suppose that m factories (called sources) supply n warehouses (called destinations) with a certain product. Factory F_i (i = 1 2, ..., m) produces a_i units (total or per unit time), and warehouse W_j (j = 1, 2, 3 ..., n) requires b_j units. Suppose that the cost of shipping from factory F_i to warehouse W_j is directly proportional to the amount shipped; and that the unit cost is c_{ij}. Let the decision variables, x_{ij}, be the amount shipped from factory F_i to warehouse W_j. The objective is to determine the number of units transported from factory F_i to warehouse W_j so that the total transportation cost $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$ is minimized. In the mean time, the supply and demand must be satisfied exactly.

Mathematically, this problem is to find x_{ij} (i = 1, 2, ..., m ; j = 1, 2, ..., n) in order to minimize the total transportation cost $z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} (c_{ij})$, subject to the restrictions of the form

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1, 2, \ldots, m \text{ (factory)}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1, 2, \ldots, n \text{ (warehouse)}$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j, \text{ and } x_{ij} \geq 0, (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n).$$

For detailed discussion, refer **chapter 10** on ***Transportation Problem.***

3. Efficiencing on Operation of System of Dams. In this problem, we determine variations in water storage of dams which generate power so as to maximize the energy obtained from the entire system. The physical limitations of storage appear as inequalities.

4. Optimum Estimation of Executive Compensation. The objective here is to determine a consistent plan of executive compensation in an industrial concern. Salary, job ranking and the amounts of each factor required on the ranked job level are taken into consideration by the constraints of linear programming.

5. Agricultural Applications. Linear programming can be applied in agricultural planning for allocating the limited resources such as acreage, labour, water, supply and working capital, etc. so as to maximize the net revenue.

6. Military Applications. These applications involve the problem of selecting an air weapon system against gurillas so as to keep them pinned down and simultaneously minimize the amount of aviation gasoline used, a variation of transportation problem that maximizes the total tonnage of bomb dropped on a set of targets, and the problem of community defence against disaster to find the number of defence units that should be used in the attack in order to provide the required level of protection at the lowest possible cost.

7. Production Management. Linear programming can be applied in production management for determining product mix, product smoothing, and assembly time-balancing.

8. Marketing Management. Linear programming helps in analysing the effectiveness of advertising compaign and time based on the available advertising media. It also helps travelling sales-man in finding the shortest route for his tour.

9. Manpower Management. Linear programming allows the personnel manager to analyse personnel policy combinations in terms of their appropriateness for maintaining a steady-state flow of people into through and out of the firm.

10. Physical Distribution. Linear programming determines the most economic and efficient manner of locating manufacturing plants and distribution centres for physical distribution.

Besids above, linear programming involves the applications in the area of administration, education, inventory control, fleet utilization, awarding contract, and capital budgeting etc.

Q. 1. Give a brief account of applications of linear programming problem.

2. Explain the meaning of a Linear Programming Problem stating its uses and give its limitations. **[C.A. (May) 95]**

3. State in brief uses of linear programming Technique.

3.12. ADVANTAGES OF LINEAR PROGARMMING TECHNIQUES

The advantages of linear programming techniques may be out-lined as follows :

1. Linear programming technique helps us in making the optimum utilization of productive resources. It also indicates how a decision maker can employ his productive factors most effectively by choosing and allocating these resources.

2. The quality of decisions may also be improved by linear programming techniques. The user of this technique becomes more objective and less subjective.

3. Linear programming technique provides practically applicable solutions since there might be other constraints operating outside the problem which must also be taken into consideration just because, so many units must be produced does not mean that all those can be sold. So the necessary modification of its mathematical solution is required for the sake of convenience to the decision maker.

4. In production processes, high lighting of bottlenecks is the most significant advantage of this technique. For example, when bottlenecks occur, some machines cannot meet the demand while others remain idle for some time.

Q. 1. What are the advantages of Linear Programming Technique ?

2. Give the properties of a Linear Programming Problem. **[JNTU (B. Tech.) 2003]**

SELF EXAMINATION PROBLEMS

1. A manufacturer of Furniture makes two products : chairs and tables. Processing of these products is done on two machines *A* and *B*. A chair requires 2 hours on machine *A* and 6 hours on macine *B*. A table requires 5 hours on machine *A* and no time on machine *B*. There are 16 hours of time per day available on machine *A* and 30 hours on machine *B*. Profit gained by manufacturer from chair and a table is Re. 1 and Rs. 5 respectively. What should be daily production of each of the two products ? **[Bikaner 92 (S)]**

   [Hint. Solve graphically. Formulation is : Max. $z = x_1 + 5x_2$, s.t. $2x_1 + 5x_2 \le 16$, $6x_1 \le 30$, and $x_1\ x_2 \ge 0$.
   Vertices of feasible region are : (0, 0), (6, 6/5), (0, 3·2)]
   [Ans. 3·2 tables, no chair, max, profit = Rs. 16].

2. The *ABC* Electric Appliance Company produces two products : Refrigerators and ranges. Production takes place in two separate departments. Refrigerators are produced in Dapartment I and ranges are produced in Department II. The company's two products are produced and sold on a weekly basis. The weekly production cannot exceed 25 regrigerators in Department I and 35 ranges in Department II, because of the limited available facilities in these two departments. The company regularly employs a total of 60 workers in the two departments. A refrigerator requires 2 man-week of labour, while a range requires 1 man-week of labour. A refrigerator contributes a profit of Rs. 60 and a range contributes a profit of Rs. 40. How many units of refrigerators and ranges should the company produce to realize a maximum profit ? **[Delhi (M. Com.) 93]**

   [Hint. Formulation : Max. $z = 60x_1 + 40x_2$; s.t. $2x_1 + x_2 \le 60$, $0 \le x_1 \le 25$, $0 \le x_2 \le 35$.
   Vertices of the feasible region are : (0, 0), (25, 0), (25, 10), (25/2, 25), (0, 35)]
   [Ans. 12·5 refrigerators, 35 ranges, max. profit z = Rs. 2150].

3. A farm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. In view of the need to ensure certain nutrient constituents, it is necessary to buy products (call them *A* and *B*) in addition. The contents of the various products, per unit, in nutrients are vitamins, proteins etc. is given in the follwoing table :

Nutrients	Nutrient Content in		Min. Amount of Nutrient
	A	B	
M_1	36	6	108
M_2	3	12	36
M_3	20	10	100

   The last column of the above table gives the minimum amount of nutrient constituents M_1, M_2, M_3 which must be given to the pigs. If the products *A* and *B* cost Rs. 20 and Rs. 4 per unit respectively, how much each of these two products should be bought so that the total cost is minimized ?

   [Hint. *Formulation* : Min. $z = 20x_1 + 40x_2$, s.t. $36x_1 + 6x_2 \ge 108$, $3x_1 + 12x_2 \ge 36$, $20x_1 + 10x_2 \ge 100$; $x_1, x_2 \ge 0$]
   [Ans. 4 units of product *A*, 2 units of product *B*, Min. Cost = Rs. 160].

4. A company produces two types of leather belts, say type *A* and *B*. Belt *A* is of superior quality and belt *B* is of a lower quality. Profits on the two types of belt are 40 and 30 paise per belt, respectively. Each belt of type *A* requires twice as much time as required by a belt of type *B*. If all belts were of type *B*, the company would produce 1,000 belts per day. But the supply of leather is sufficient only for 800 per day. Belt *A* requires a fancy buckle and 400 fancy buckles are available for this, per day. For belt of type *B*, only 700 buckles are available per day. How should the company manufacture the two types of belt in order to have maximum overall profit ? **[Jodhpur 93]**

   [Hint. Formulation is : Max. $z = 0{\cdot}40x_1 + 0{\cdot}30x_2$, s.t. $x_1 + x_2 \le 800$. $2x_1 + x_2 \le 1000$, $0 \le x_1 \le 400$, $0 \le x_2 \le 700$.
   Graphically. Vertices of feasible region are : (0, 0), (400, 0), (400, 200), (200, 600), (100, 700), (0, 700)].
   [Ans. 200 belts of type *A*, 600 belts of type *B*, max. profit = Rs. 260].

5. A company sells two different products *A* and *B*. The company makes a profit of Rs. 40 and Rs. 30 per unit on products *A* and *B* respectively. The two products are produced in a common production process and are sold in two different markets. The production process has a capacity of 30,000 man-hours. It takes 3 hours to produce one unit of *A* and one hour to produce one unit of *B*. The market has been surveyed and company officials feel that the maximum number of units of *A* that can be sold is 8,000 and the maximum of *B* is 12,000 units. Subject to these limitations, the products can be sold in any convex combinations.
   Formulate the above problem as a L.P.P. and solve it by graphical method.

   [Hint. Formulation of the problem is : Max. $z = 40x_1 + 30x_2$, *s.t.*, $3x_1 + x_2 \le 30{,}000$, $0 \le x_1 \le 8{,}000$, $0 \le x_2 \le 12{,}000$.
   Vertices of the feasible region are : (0, 0), (8000, 0), (8000, 6000), (6000, 12000), (0, 12000)].
   [Ans. x_1 = 6000 units of A, x_2 = 12000 units of B, max. z = Rs. 6,00,000].

6. A person require 10, 12 and 12 units of chemicals *A*, *B* and *C* respectively for his garden. A liquid product contains 5, 2 and one units of *A*, *B* and *C* respectively per jar. A dry product contains, 1, 2 and 4 units of *A*, *B* and *C* per carton. If the liquid product sells for Rs. 3 per jar and the dry product sells for Rs. 2 per carton, how many of each should be purchased to minimize the cost and meet the requirements. **[Banasthali (M.Sc.) 93]**

   [Hint. Formulation : Min. $z = 3x_1 + 2x_2$, *s.t.* $5x_1 + x_2 \ge 10$, $2x_1 + 2x_2 \ge 12$, $x_1 + 4x_2 \ge 12$ and $x_1, x_2 \ge 0$.
   The vertices of the feasible region are : (12, 0), (4, 2), (1, 5), (0, 10)].
   [Ans. x_1 = 1 unit of liquid product, x_2 = 5 units of dry product, min. cost of Rs. 13].

7. An automobile manufacturer makes automobiles and trucks in a factory that is divided into two shops. Shop A, which perform the basic assembly operation must work 5 man-days on each truck but only 2 man-days on each automobile. Shop B, which performs finishing operations must work 3 man-days for each automobile or truck that it produces. Because of men and machine limitations shop A has 180 man-days per week available while shop B has 135 man-days per week. If the manufacturer makes a profit of Rs. 300 on each truck and Rs. 200 on each automobile, how many of each should be produced to maximize his profit ?
[Hint. Formulation is : Max. $\mathbf{z} = 300x_1 + 200x_2$; *s.t.* $5x_1 + 2x_2 \le 180$, $3x_1 + 3x_2 \le 135$, $x_1, x_2 \ge 0$]
[Ans. $x_1 = 30$ trucks, $x_2 = 15$ automobiles per week, max. $\mathbf{z}$ = Rs. 12,000].

8. The manager of an oil refinary must decide on the optimal mix of two possible blanding processes of which the inputs and outputs per production run are as follows :

Process	Input (Units)		Output (Units)	
	Crude A	Crude B	Gasoline X	Gasoline Y
1	5	3	5	8
2	4	5	4	4

The maximum amounts available of crude A and B are 200 units and 150 units respectively. Market requirements show that at least 100 units of gasoline X and 80 units of gasoline Y must be produced. The profits per production run from process 1 and process 2 are Rs. 300 and Rs. 400 respectively. Solve the LP problem by graphical method.

[Gujarat (M.B.A) 1998]

[Hint. Formulation is : Max. $\mathbf{z} = 300x_1 + 400x_2$, *s.t.*
$5x_1 + 4x_2 \le 200$, $3x_1 + 5x_2 \le 150$, $5x_1 + 4x_2 \ge 100$, $8x_1 + 4x_2 \ge 80$, $x_1, x_2 \ge 0$.
The vertices of the feasible region are : (20, 0), (40, 0), (400/13, 150/13), (0, 30), (0, 25)].
[**Ans.** $x_1 = 400/13$, $x_2 = 150/13$, Max. $\mathbf{z} = 1{,}80{,}000/13$].

9. A manufacturer makes two products P_1 and P_1 using two machines M_1 and M_2. Product P_1 requires 2 hours on machine M_1 and 6 hours on machine M_2. Product P_2 requires 5 hours on machine M_1 and no time on machine M_2. There are 16 hours of time per day available on machine M_1 and 30 hours on M_2. Profit margin from P_1 and P_2 is Rs. 2 and Rs. 10 per unit respectively. What should be the daily production mix to optimize profit ?
[Ans. $P_1 = 3{\cdot}2$, $P_2 = 0$, max profit = Rs. 16.]

10. Upon completing the construction of his house Dr. Sharma discovers that 100 square feet of plywood scrap and 80 square feet of white-pine scrap are in usable form for the construction of tables and book cases. It takes 16 square feet of plywood and 8 square feet of white-pine to make a table; 12 square feet of plywood and 16 square feet of white-pine are required to construct a book case. By selling the finished products to a local furniture store, Dr. Sharma can realise a profit of Rs. 25 on each table and Rs. 290 on each book case. How may he most profitably use the left-over wood ? Use graphical method to solve the problem.
[Hint. Formulation of this problem is :
Max. $\mathbf{z} = 25x_1 + 20x_2$, subject to the constraints : $16x_1 + 12x_2 \le 100$, $8x_1 + 16x_2 \le 80$; $x_1, x_2 \ge 0$].
[Ans. 4 tables, 3 book-cases, max. profit = Rs. 160].

11. A television company has three major departments for manufacture of its two models, A and B. Monthly capacities are given as follows :

Departments	Per unit time requirements (hours)		Hours available this month
	Model A	Model B	
I	4·0	2·0	1600
II	2·5	1·0	1020
III	4·5	1·5	1600

The marginal profit of model A is Rs. 400 each and that of model B is Rs. 100 each. Assuming that the company can sell any quantity of either product due to favourable market conditions, determine the optimum out-put for both the models, the highest possible profit for this month and the slack time in the three departments.
[Hint. Formulation of the problem is : Max, $\mathbf{z} = 400x_1 + 100x_2$, subject to
$4x_1 + 2x_2 \le 1600$, $2{\cdot}5x_1 + x_2 \le 1200$, $4{\cdot}5x_1 + 1{\cdot}5\, x_2 \le 1600$; $x_1, x_2 \ge 0$]
[Ans. $x_1 = 3200/9$, $x_2 = 0$, max. $\mathbf{z} = 128000/9$].

12. A caterer knows that he will need 40 napkins on a given day and 70 napkins the day after. He can purchase napkins at 20 paise each and, after they are purchased, he can have dirty napkins laundered at 5 paise each for using the next dary. In order to minimize his costs, how many napkins should he purchase initially and how many dirty napkins should have laundered.
[Hint. Formulation of the problem is : Max. $\mathbf{z} = 0{\cdot}20x_1 + 0{\cdot}05x_2$, subject to $x_1 \ge 70$, $x_2 \ge 40$, $x_1, x_2 \ge 0$]
[Ans. The caterer should buy 70 napkins and have 40 laundered after the first day].

13. A salesman handles two products. He does not expect to sell more than 10 units/month of product A or 39 units/month of product B. To avoid a penalty he must sell at least 24 units of product B. He receives a commission of 10 per cent on all sales, and he must pay own costs, which are estimated at Rs. 1·50 per hour spent on making calls, he works only part

time and therefore is willing to work a maximum of 80 hours/month. Product *A* sells for Rs. 150 per unit and requires an average of 1·5 hours on every call; the probability of making a sale is 0·5. Product *B* sells for Rs. 70 per unit and requires an average of 30 minutes on every call; the probability of making a sale is 0·6.

How many calls per month should he make on customers for each product.

14. The *ABC* company wishes to plan its advertising strategy. There are two media under consideration, call them magazine I and magazine II. Magazine I has a reach of 2,000 potential customers and magazine II has a reach of 3,000 potential customers. The cost per page of advertising is Rs. 400 and Rs. 600 for magazines I and II respectively. The firm has a monthly budget of Rs. 6,000. There is an important requirement that the total reach for the income group under Rs. 20,000 per annum, should not exceed 4,000 potential customers. The reach in magazine I and magazine II for this income group is 400 and 200 potential customers. How many pages should be brought in the two magazines to maximize the total reach. Solve it graphically. **[AIMS (BE Ind.) Bang. 2002]**

15. A publisher of text books is in the process of presenting a new book to be marketed. The book may be bound by either cloth or hard paper. Each cloth bound book sold contributed Rs. 30, and each paper bound book contributed Rs. 28. It takes 10 minutes to bind a cloth cover, and 9 minutes to bind a paper-back. The total available time for binding is 800 hours. After considerable market survery, it is predicted that the cloth cover sales will exceed at least 10,000 copies, but the paper-back sales will be no more than 6000 copies.

Draw the feasible region which corresponds graphically to each constraint, and find the optimal solution.

16. A poultry producer must feed his stock daily at least 124 units of nutritional element *A* and 60 units of nutritional element *B*. He has available two feeds. One kg. of feed 1 costs Rs. 16 and contains 10 units of *A* and 3 units of *B*. One kg. of feed 2 costs Rs. 14 and contains 4 units of *A* and 5 units of *B*. Determine least expensive adequate diet, by graphical method.

17. The *ABC* company has been a producer of picture tubes to telivision sets and certain printed circuits for radio. The company has just expanded into full scale production and marketing of *AM* and *AM–FM* radio. It has built a new plant that can operate 48 hours per week. Production of an *AM* radio in the new plant will require 2 hours and production of an *AM–FM* radio will require 3 hours. Each *AM* radio will contribute Rs. 40/- to profits while an *AM – FM* radio will contribute Rs. 80/- to profits. The marketing department after extensive research, has determined that a maximum of 15 *AM* radios and, 10 *AM–FM* radios can be sold each week.

(i) Formulate a linear programming model to determine the optimumn production mix of *AM* and *AM–FM* radio that will maximize profits.

(ii) Solve the above problem using graphical method. **[Delhi (M.B.A) Nov. 98]**

[**Hint.** The formulation is : Max. $z = 40x_1 + 80x_2$, s.t $2x_1 + 3x_2 \le 48$, $x_1 \le 15$, $x_2 \ge 10$, $x_1 \ge 0$, $x_2 \ge 0$]

[**Ans.** $x_1 = 9$, $x_2 = 10$ and max $z = 1{,}160$]

18. The *ABC* clothing stores is making plans for its annual shirt and pants sale. The owner Mr. Jain is planning to use two different types of advertizing, viz. radio and newspaper ads to promote the sales. Based on past experience, Mr. Jain feels confident that each newspaper ad will reach 40 shirt customers and 80 pant customers. Each radio ad will reach 30 shirt-customers and 20 pant-customers, he feels. The cost of each newspaper ad is Rs. 300 and the cost of each radio spot is Rs. 450. An advertizing agency will prepare the advertizing and it will require 5 man-hours of preparation for each newspaper ad and 15 man-hours of preparation for each radio spot. Mr. Jain's sales manager says that a minimum of 75 man-hours should be spent on preparation of advertizing in order to fully utilize the service of advertizing agency. Mr. Jain feels that in order to have a successful sale the advertizing must reach at least 360 shirt costomers and at least 400 pant customers.

(i) Formulate a linear programming model to determine how much advertizing should be done using each media in order to minimize costs and still attain the objectives of Mr. Jain.

(ii) Solve the above problem by graphical method.

19. A city police department has the following minimal daily requirement for police officers during the six shift periods :

Time of Day :	2 *a.m.*–6 *a.m.*	6 *a.m.*–10 *a.m.*	10 *a.m.*–2 *p.m.*	2 *p.m.*–6 *p.m.*	6 *p.m.*–10 *p.m.*	10 *p.m.*–2 *a.m.*
Peirod :	1	2	3	4	5	6
Minimal No. Required :	22	55	88	110	44	33

An officer must start at the beginning of a 4 hour stift and stay on duty for two consecutive shirts (an 8-hour four). Any one starting during 6 stays on duty during period 1 of the next day. The objective of the police department is to always have on duty the minimal number required in a period but to so with the least number of officers. Develop the corresponding linear programming model. **[IAS (Maths.) 99]**

20. You are operating Narula's drive-in franchise and are contemplating trying an exprimental 24-hour operation. History of previous operations and estimates indicate that the following number of counter (clerks) will be needed

Time of day :	6–0	10–14	14–18	18–22	22–02	02–06
Counter clerks required :	8	11	7	14	5	4

Each clerk works on eight consecutive hour shift. How would you schedule the clerks to satisfy these requirements while minimizing your daily clerk force ? Formulate this problem as linear programming problem. **[Delhi (MBA) April 92]**

21. A manufacturer produces two different models, *X* and *Y* of the same product. The raw materials r_1, r_2 are required for production. At least 18 kg of r_1, and 12 kg of r_2 must be used daily. Also, at most 34 hours of labour are to be utilised. 2 kg of r_1' are needed for each model *X* and 1 kg of r_1 for each model *Y*. For each model of *X* and *Y*, 1 kg of r_2 is required. It takes 3 hours to manufacture a model X and 2 hours to manufacture a model Y. The profit is Rs. 50 for each model X and Rs. 30 for each model Y. How many units of each model should be produced to maximize the profit.

[**Hint.** The formulation of the problem is; Max $z = 50x_1 + 30x_2$, st. $2x_1 + x_2 \geq 18$, $x_1 + x_2 \geq 12$, $3x_1 + 2x_2 \leq 34$, $x_1 \geq 0$, $x_2 \geq 0$.

Ans. $x_1 = 10$, $x_2 = 2$ and max $z = 560$.]

22. An investor has money making activities A_1, A_2, A_3 and A_4. He has only one lakh rupees to invest. In order to avoid excessive investment, no more than 50% of the total investment can be placed in activity A_2 and/or activity A_3. Activity A_1 is very conservative, while activity A_4 is speculative. To avoid excessive speculation at least Re. 1 must be invested in activity A_1 for every Rs. 3 invested in activity A_4. The data on the return on investment are as follows :

Activity	Anticipated Return on Investment (%)
A_1	10
A_2	12
A_3	14
A_4	16

The investor wishes to know how much to invest in each activity to maximize the total return on the investment.
[Delhi (MBA) Nov. 96]

23. Let us assume that you have inherited Rs. 1,00,000 from your father-in-law that can be invested in a combination of two stock portfolios, with the maximum investment allowed in either portfolio set at Rs. 75,000. The first portfolio has an average rate of return of 10%, whereas the second has 20%. In terms of risk factors associated with these portfolios, the first has a risk rating of 4 (on a scale from 0 to 10), and the second has 9. Since you wish to maximize your return you will not accept an average rate of return below 12% or a risk factor above 6. Hence you then face the important question. How much should you invest in each portfolio ? Formulate as a linear programming problem and solve it by graphical method. **[C.A. (May) 99]**

24. A company need 50 new machines. The machines have an economic life of two years and can be purchased for Rs. 4,500 or leased for Rs. 2,800 per year. The purchased machines at the end of two years, have no salvage value. Company has Rs. 1,00,000 in uncommitted funds that can be used for the purchase or lease of machines at the beginning of year 1. The company can obtain a loan of upto Rs. 2,00,000 at 18 per cent interest per year. According to the terms of the loan, the company has to repay the amount borowed plus the interest at the end of each year. Each machine can earn Rs. 3,000 per year. The earnings from the first year can be used to lease costs and the repayment of debts at the start of second year. The company wants to minimize the total cost of purchasing machines or leasing machines during the years 1 and 2 and to minimize the interest payment on funds borrowed to obtain the machines. Formulate this problem as a linear programming problem. **[Delhi (MBA) 98]**

25. The *PQR* Stone company sells stone secured from any of three adjacent quarries. The stone sold by the company must conform to the following specifications :

Material X equal to 30%,
Material Y equal to or less than 40%
Material Z between 30% and 40%.

Stone from quarry A costs Rs. 10 per tonne and has the following properties :
Material X—20%; Material Y—60% and Material Z—20%.

Stone from quarry B costs Rs. 12 per tonne and has the following properties :
Material X—40%; Material Y—30% and Material Z—30%

Stone from quarry C costs Rs. 15 per tonne and has the following properties :
Material X—10%; Material Y—40% and Material Z—50%.

Formulate the above as a linear programming problem to minimize cost per tonne. **(Delhi (M.B.A.) April 99)**

[**Hint.** The data can be summarized as follows :

	A	B	C	
X	20%	40%	10%	= 30%
Y	60%	30%	40%	≤ 40%
Z	30%	30%	50%	between 30% & 40%
Costs	10	12	15	

The problem is : Min. $z = 10x_1 + 12x_2 + 15x_3$

s.t. $2x_1 + 4x_2 + x_3 = 3$, $6x_1 + 3x_2 + 5x_3 \leq 4$, $2x_1 + 3x_2 + 5x_3 \leq 4$,

$2x_1 + 3x_2 + 5x_3 \geq 3$; x_1 , x_2 , $x_3 \geq 0$.]

26. Formulate a linear programming model for the following inventory and work force balance problem :

Your production requirements in each of the next five months are :

12,000 15,000 10,000 14,000 6,000